Contents 34

Modelling Motion

Learning outcomes

This Workbook follows on from Workbook 5 in describing ways in which mathematical techniques are used in modelling. In this Workbook you will learn how use of vectors provides shorthand descriptions of projectile motion in several contexts, motion in a circle and on curved paths such as in fairground rides. Also you will learn how the complication of velocity-dependent resistance to motion can be handled in certain cases.

Projectiles

34.1

Introduction

In this Section we study the motion of projectiles constrained only by gravity. Although historically the mechanics of projectile motion were studied and developed mainly in military contexts, there are many relevant non-military situations. For example botanists study the mechanics of dispersal of seeds from exploding pods; hydraulic engineers are interested in the distribution and settling of sediments and particles; many athletic activities and sports such as skiing and diving involve humans acting as projectiles through leaping or hurdling or otherwise throwing themselves about. Other sporting activities involve inanimate projectiles e.g. balls of various kinds, javelins. Precise models of some possible situations, for example swerving or swinging or spinning balls, or ski-jumping involve rather complicated kinds of motion and require considerations of resistive forces and aerodynamic forces. First trips around the modelling cycle (see HELM 5), sometimes second trips, are given here.

Prerequisites

Before starting this Section you should ...

- be able to use vectors and to carry out scalar and vector products
- be able to use Newton's laws to describe and model the motion of particles
- be able to use coordinate geometry to study circles and parabolas
- be able to use calculus to differentiate and integrate polynomials

Learning Outcomes

On completion you should be able to ...

- use vector notation to represent the position, velocity and acceleration of projectiles, objects moving on inclined planes and objects moving on curved paths
- compute frictional forces on static and moving objects on inclined planes and on objects moving at constant speed around bends

1. Introduction

In this Section we study the motion of projectiles constrained only by gravity. We revise the model, based on Newtons laws, for the motion of an object falling vertically without air resistance and extend this to two dimensions using vector functions to represent position, velocity and acceleration. It is pointed out that an object falling under gravity or thrown vertically upwards before falling back under gravity are simple examples of projectiles. More interesting projectiles involve horizontal as well as vertical motion. The vector nature of the motion is explored. The influences of launch height and launch angle are explored in various contexts. Also we consider the motion of objects constrained to move on inclined planes (e.g. the balls in pinball machines).

2. Projectiles: an introduction

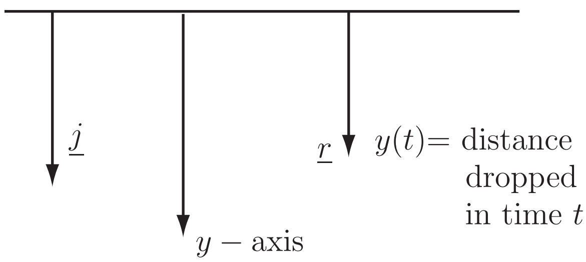
Vertical motion under gravity

Consider a marble which is thrown horizontally off the Clifton Suspension Bridge at a speed of 10 m s^{-1} and falls into the River Avon. We wish to find the location at which it will splash into the river. We assume that the only force acting on the marble is the force of gravity and that this force is constant. The marble is regarded as a projectile i.e. a point object which has mass but does not spin or rotate. Another assumption made is that the Earth is locally flat. Since the initial vertical speed is zero, application of the distance-time equation ($s = ut + \frac{1}{2}at^2$) to the vertical motion gives

$$y(t) = \frac{1}{2}gt^2 \qquad (1.1)$$

where y is measured downwards from the bridge and g is the acceleration due to gravity.
The position vector of an object falling freely in the vertical ($\underline{j}$) direction with zero initial velocity and no air resistance may be expressed as a position vector $\underline{r}(t)$ which is a variable vector depending on the (scalar) variable t representing the time where

$\underline{r}(t) = y(t)\underline{j} = \frac{1}{2}gt^2\underline{j}$, illustrated in the diagram below.



For motion in a straight line there is no particular reason for introducing vectors. However, time-dependent vectors may be used to describe more complicated motion - for example that along curved paths. By introducing the horizontal unit vector $\underline{i}$ in addition to the vertical unit vector $\underline{j}$, a position vector in two dimensions may be written

$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j}$.

For an object falling **vertically**, $x(t) = 0$ because x does not change with time. Suppose, however, that the object were to have been launched **horizontally** at speed u. Then, if air resistance is ignored and there are no other forces acting in the horizontal direction, the horizontal acceleration is zero and the horizontal speed of the object should remain constant. This means that the horizontal coordinate is given by $x(t) = ut$ and, using the earlier result for $y(t)$, the vector function describing the position at time t of an object thrown horizontally from some point, which is taken as the origin of coordinates, is given by

$$\underline{r}(t) = ut\underline{i} + \frac{1}{2}gt^2\underline{j}. \qquad (1.2)$$

The coordinate system and the vectors corresponding to such a situation are shown in Figure 1.

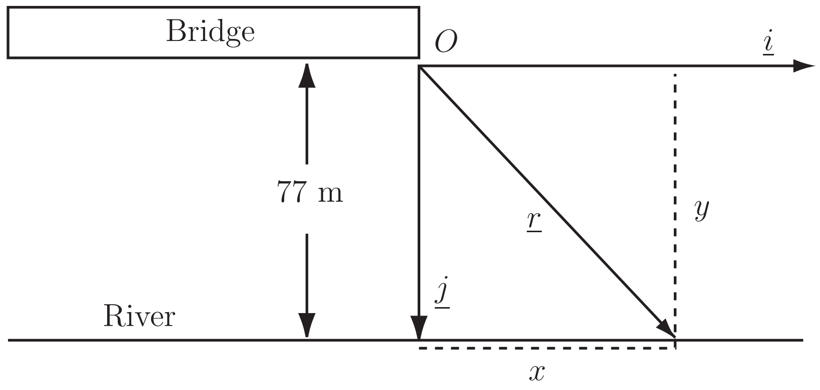


Figure 1: Coordinate system and unit vectors for an object thrown from a bridge

The information in Equation (1.2) is sufficient to determine the object's position graphically at any time t, since it gives both $x(t)$ and $y(t)$ and hence it is possible to plot $y(t)$ against $x(t)$ for various values of t. An example calculation of the path during the first 3.5 s of the descent of an object thrown horizontally at 10 m s^{-1} with $g = 9.81$ m s^{-2} is shown in Figure 2. In this graph values of $y(t)$ increase downward so that the curve corresponds to the downward path of the object. The technical name used to describe such a path is the **trajectory**.

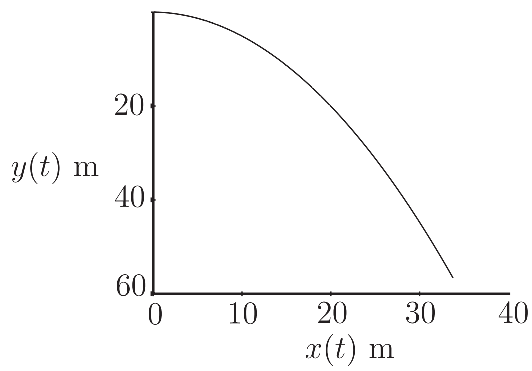


Figure 2

Trajectory for first 3.5 s of an object thrown horizontally from a bridge at 10 m s^{-1} ignoring air resistance

Given the vector components of the time-dependent velocity, it is possible to calculate its magnitude and direction at any time. The magnitude is given by the square root of the sum of the squares of the components. Hence, from the last example, the magnitude of the position vector is given by

$$|\underline{r}| = \left(u^2t^2 + \frac{1}{4}g^2t^4\right)^{1/2}.$$

The angle of the position vector measured clockwise from the x-direction is given by

$$\cos\phi = ut/|\underline{r}| \qquad \sin\phi = \frac{1}{2}gt^2/|\underline{r}|$$

so

$$\tan \phi = \frac{1}{2} gt/u.$$

Note that the angle is zero when t is zero and increases with t (as might be expected). Figure 3 shows graphs of $|\underline{r}|$ and ϕ for the example and values of t considered in Figure 2.

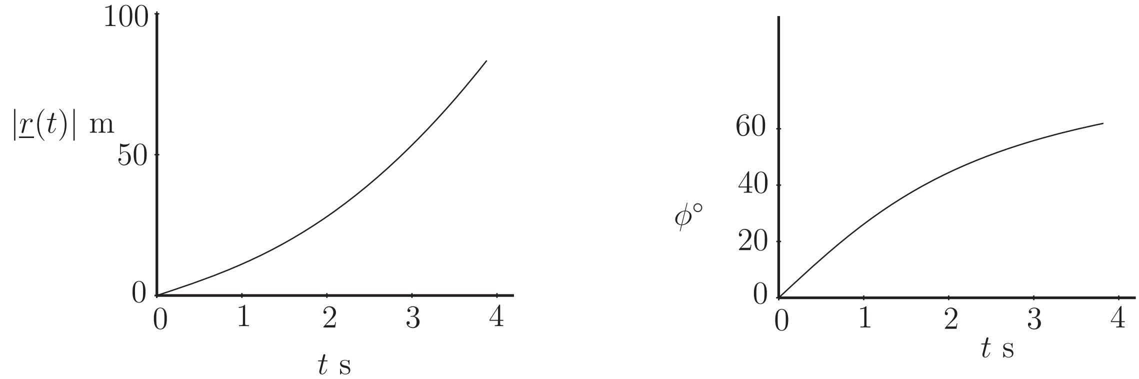


Figure 3: Values of $|\underline{r}|$ and $\phi^\circ (= \phi(t) \times (180/\pi))$ for the object projected from a bridge

Note that ϕ is the angle that the position vector makes with the horizontal and does not denote the direction of motion (i.e. the velocity) of the object. Note also that by introducing another unit vector $\underline{k}$ at right-angles to both $\underline{i}$ and $\underline{j}$ it is possible to consider motion in three dimensions.

Task

Write down the position vector for a particle moving so that its coordinates are given by

$$x = 2\cos(wt) \qquad y = 2\sin(wt) \qquad z = 1.$$

What is the corresponding magnitude of this vector? How would you describe the resulting motion?

Your solution

Answer

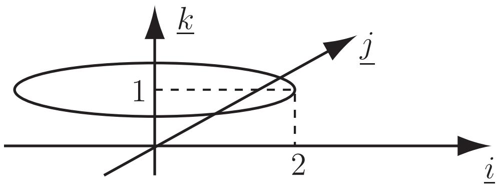
The position vector may be written

$$\underline{r}(t) = 2\cos(\omega t)\underline{i} + 2\sin(\omega t)\underline{j} + \underline{k}.$$

Hence

$$|\underline{r}(t)| = \sqrt{4\cos^2(\omega t) + 4\sin^2(\omega t) + 1} = \sqrt{5}.$$

Since this is constant, the particle stays at a constant distance from the origin during its motion. When $t = 0,\ \underline{r}(0) = 2\underline{i} + \underline{k}$.



The object is moving in a circle of radius 2 in the $z = 1$ plane (see diagram).

Show that the vector function

$$\underline{r}(t) = at\underline{i} + bt^2\underline{j},$$

where a and b are constant scalars, can be represented by a parabola.

By comparing Equation (1.1) with the equation in this Task demonstrate that the trajectory shown in Figure 2 is part of a parabola.

Your solution

Answer

Given $\underline{r}(t) = at\underline{i} + bt^2\underline{j} = x(t)\underline{i} + y(t)\underline{j}$ we can write $x(t) = at$ and $y(t) = bt^2$. Using the first of these to obtain $t = \dfrac{x}{a}$ and substituting for t in the second, we obtain $y = b\dfrac{x^2}{a^2} = cx^2$ where c is a constant. This has the form of a parabola centred on $(0, 0)$.

Suppose that we wish to calculate the coordinates at which the marble will splash into the River Avon, given that the water surface is 77 m below the point of launch. Since the horizontal component of velocity is not changing during the fall, we concentrate on the vertical motion. The strategy is to calculate the length of time it takes to drop through the vertical distance between the point of launch and the water surface and then use this time to calculate the horizontal distance moved at constant speed. We use Equation (1.1) to calculate the length of time needed to fall 77 m i.e. the value of t such that

$$\frac{1}{2}gt^2 = 77.$$

This gives $t = \sqrt{\dfrac{2 \times 77}{9.81}} = 3.96$ s. During this time the marble will have moved a horizontal distance ut. So if $u = 10$ m s^{-1}, the horizontal distance moved is 39.6 m and the coordinates of the splash down are (39.6, 77.0).

The question arises of how to deal with more general problems of a similar nature but starting from first principles. This question leads to a fuller consideration of vector representations of motion.

Velocity and acceleration vectors

The first derivative of time-dependent position vectors may be identified as the velocity vector and the second derivative as the acceleration vector. So, for the example of a stone falling from rest under gravity without air resistance, given that the velocity vector is the first derivative of the position vector,

$$\underline{v}(t) = \frac{d}{dt}\underline{r}(t) = \frac{d}{dt}(\frac{1}{2}gt^2\underline{j}) = gt\underline{j} \quad \text{(since } \underline{j} \text{ does not vary with } t\text{)}.$$

Similarly, the acceleration vector is the second derivative of the position vector, which will be the same as the first derivative of the velocity vector, so

$$\underline{a}(t) = \frac{d}{dt}\underline{v}(t) = \frac{d^2}{dt^2}\underline{r}(t) = g\underline{j}.$$

Note that this is an expected result (the acceleration is that due to gravity).

In two dimensions

$$\underline{v}(t) = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j}$$

and

$$\underline{a}(t) = \frac{d^2x}{dt^2}\underline{i} + \frac{d^2y}{dt^2}\underline{j}.$$

For the marble thrown horizontally at velocity u from the bridge

$$\underline{v}(t) = u\underline{i} + gt\underline{j}$$

and

$$\underline{a}(t) = g\underline{j}.$$

Note that the horizontal and vertical parts of the velocity (or acceleration) are called the horizontal and vertical **components** respectively. For the marble thrown horizontally at speed u from the bridge the horizontal component of velocity at any time t is u and the vertical component of velocity at any time t is gt.

Since each component of the vector is differentiated separately, the integral of the acceleration vector may be identified with a velocity vector and the integral of the velocity vector may be identified with a position vector. These give the same expressions as those that we started with apart from arbitrary constants. Note that when integrating *vector* expressions the arbitrary constant is a constant *vector*.

Example 1

(a) Use integration and the variables and vectors identified in Figure 1 to derive vector expressions for the velocity and position of an object thrown horizontally from a bridge at speed u ignoring air resistance.

(b) Find the object's coordinates after it has dropped a distance h.

Solution

(a) The acceleration has only a vertical component i.e. the acceleration due to gravity. $\underline{a}(t) = g\underline{j}$.

Integrating once gives $\underline{v}(t) = gt\underline{j} + \underline{c}$ where $\underline{c}$ is a constant vector.

The initial velocity has only a horizontal component, so $\underline{v}(0) = \underline{c} = u\underline{i}$ and $\underline{v}(t) = u\underline{i} + gt\underline{j}$.

Integrating again $\underline{r}(t) = ut\underline{i} + \dfrac{1}{2}gt^2\underline{j} + \underline{d}$ where $\underline{d}$ is another constant vector.

Since $\underline{r}(0) = \underline{0}$, then $\underline{d} = \underline{0}$ (the zero vector), so

$$\underline{r}(t) = ut\underline{i} + \frac{1}{2}gt^2\underline{j}$$

which is the result obtained previously as Equation (1.2) by considering the horizontal and vertical components of motion separately.

(b) The position coordinates at any time t are $\left(ut, \dfrac{1}{2}gt^2\right)$.

When $y(t) = h$, then $\frac{1}{2}gt^2 = h$, or

$$t = \sqrt{\frac{2h}{g}}. \qquad (1.3)$$

At this value of t, $x(t) = ut = u\sqrt{\dfrac{2h}{g}}$. So the coordinates when $y(t) = h$ are $\left(u\sqrt{\dfrac{2h}{g}}, h\right)$.

In this Workbook you will only meet straightforward examples of vector integration where the integral of the vector is obtained by integrating its components. More complicated vector integrals called line integrals are introduced in HELM 29.

3. Projectiles

Horizontal launches

Let us reflect on what has been done in the last example because it illustrates both the features of projectile motion in the absence of air resistance and a procedure for solving mechanics problems involving projectiles. Instead of the vector method used in Example 1, the relevant projectile motion could have been considered in terms of the separate equations of motion in the horizontal (x-) and vertical (y-) directions; these may be written

$$\ddot{x} = 0 \qquad \ddot{y} = g.$$

These may be solved separately but the vector method is neater since it shows horizontal and vertical component information at the same time.

The most important features of projectile motion in the absence of air resistance are the constant vertical acceleration and the constant horizontal speed. In projectile problems, the usual procedure is to find the time taken to reach the vertical coordinate position of interest and then to use this time together with the horizontal component of velocity to get the horizontal distance.

Example 2

In an apparatus to demonstrate two-dimensional projectile motion, ball bearings are released simultaneously to roll on two identical ramps that are separated vertically. The ramps consist of sloped and horizontal portions of the same length. The ball bearing on the upper ramp becomes a projectile when it reaches the end of the upper ramp while the lower ball bearing rolls along a horizontal channel when it reaches the end of its ramp. The situation is represented in Figure 4.

(a) What is the speed of each ball bearing at the end of its ramp (A or B in Figure 4)?

(b) How does the point at which the upper ball bearing hits the lower channel vary with the height of the upper ramp?

(c) Where will be the location of the lower ball bearing at the time at which the projectile ball bearing hits the lower channel?

(d) What assumptions have been made in answering (a), (b) and (c)?

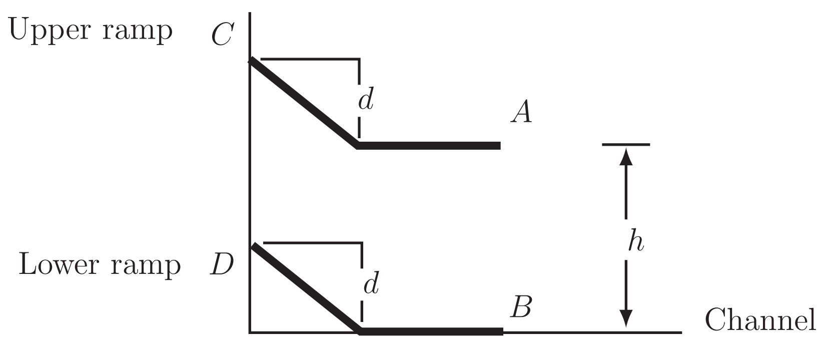


Figure 4: Side view of the projectile demonstrator

Solution

(a) The concepts of kinetic and potential energy and conservation of total energy may be used. At the top of its ramp, each ball bearing will have a potential energy with respect to the bottom of mgd, where m is its mass, g is gravity and d is the vertical drop from top to bottom of the ramp. Also it will have zero kinetic energy since it is stationary. At the bottom of the ramp, the potential energy will be zero (as long as the thickness of the ramp is ignored) and the kinetic energy will be $\frac{1}{2}mu^2$ where u is the magnitude of the velocity at the bottom of each ramp. So, by conservation of energy,

$$mgd = \frac{1}{2}mu^2, \quad \text{or} \quad u = \sqrt{2gd}.$$

This will be the component of velocity in the direction of the sloping part of the ramp and, in the absence of any losses along the ramp or at the bend where there is a sudden change in momentum, this becomes the horizontal component of velocity at the end of the ramp.

(b) Since the ramps are identical, both ball bearings will have the same horizontal component of velocity at the ends of their ramps. Suppose that we use coordinates x (horizontal) and y (downward vertical) with the origin at A. The answer to Example 2(b) may be employed without having to start from scratch. This tells us that the coordinates of the projectile ball bearing when $y = h$ are $\left(u\sqrt{\frac{2h}{g}}, h\right)$ or, since $u = \sqrt{2gd}$, the coordinates are $(2\sqrt{hd}, h)$. Since d is constant, this means that the location of the point at which the projectile ball bearing hits the lower channel varies with the square root of the height of A above B (i.e. with $\sqrt{h}$).

(c) As remarked earlier, the lower ball bearing will have the same horizontal component of velocity (u) at B as the projectile ball bearing has at A. Consequently it will travel the same horizontal distance in the same time as the projectile ball bearing. This means that the projectile ball bearing should hit the lower one.

(d) In (a) the thickness of the ramps has been ignored and the bends in the ramps have been assumed not to introduce any energy losses. In (b) air resistance has been assumed to be negligible. In (a) and (c) rolling friction along the sloping ramp and the horizontal channel has been assumed to be negligible. In fact the effects due to the bends in the ramps will mean that the calculation of horizontal velocity at the end of the ramp is not accurate. However, it can be assumed that identical bends will affect identical ball bearings identically. So the conclusion that the ball from the upper ramp will hit the lower one is still valid (provided rolling friction for the lower ball bearing is comparable to air resistance for the upper ball bearing).

Task

A crashed car is found on the beach near an unfenced part of sea wall where the top of the wall is 18 m above the beach and the beach is level. The investigating police officer finds that the marks in the beach resulting from the car's impact with the beach begin at 8 m from the wall and that the vehicle appears to have been travelling at right-angles to the wall. Estimate how fast the vehicle must have been travelling when it went over the wall.

Your solution

Answer

Use y measured downwards as the vertical coordinate. The vector equation of motion is

$$\underline{a} = g\underline{j}.$$

Integrating once gives

$$\underline{v} = gt\underline{j} + \underline{c}.$$

The car's initial vertical velocity component may be assumed to be zero. If the initial horizontal component is represented by $u\underline{i}$, then $c = u\underline{i}$ and

$$\underline{v} = u\underline{i} + gt\underline{j}.$$

Integrating again to get position as a function of time,

$$\underline{r} = ut\underline{i} + \frac{1}{2}gt^2\underline{j} + \underline{d}.$$

In accordance with the initial condition that the vertical position is measured from the top of the sea wall, $\underline{d} = \underline{0}$ and

$$\underline{r} = ut\underline{i} + \frac{1}{2}gt^2\underline{j}.$$

Now consider *vertical motion only*. When $y = 18.0$,

$$t = \sqrt{\frac{2 \times 18}{9.81}} = 1.916.$$

Answer
So the car is predicted to hit the beach after 1.916 s. Next consider *horizontal motion*. During 1.916 s, in the absence of air resistance, the car is predicted to move a horizontal distance of $u \times 1.916$. This distance is given as 8 m. So

$$8 = 1.916u$$

or

$$u = 4.175.$$

So the car is estimated to have left the sea wall at a speed of just over 4 m s^{-1} (about 15 kph).

There are several complications that may arise when studying and modelling projectile motion. Launch at some angle other than horizontal is the main consideration in the remainder of this Section. For a given launch speed it is possible to find more than one trajectory that can pass through the same target location. Another complication results from launch at a location that is not the origin of the coordinate system used for modelling the motion.

Angled launches

Vector equations may be used to obtain the position and velocity of a projectile as a function of time if the object is not launched horizontally but with some arbitrary velocity. We shall start by modelling an angled launch from and to a horizontal ground plane. Again it is sensible to use the launch point as the origin of the coordinate system employed. Ignoring air resistance, we shall find expressions for the velocity and position vectors at time t of an object that is launched from ground level ($\underline{r} = \underline{0}$) at velocity $\underline{u}$ with direction θ above the horizontal. Subsequently we shall find an expression for the time at which the object will hit the ground and the horizontal distance it will have travelled in this time. Finally we will find the coordinates of the highest point on its trajectory (see Figure 5), the angle that will give maximum range and an expression for maximum range in terms of the magnitude and angle of the launch velocity.

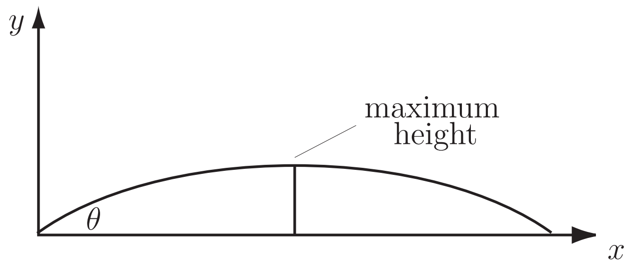


Figure 5: Path of a projectile after an angled launch from ground level

We use an upward-pointing vertical vector in all of the remaining projectile problems in this Workbook. We start from first principles using Newton's second law in vector form:

$$\underline{F} = m\underline{a}. \qquad (1.4)$$

This time, since the initial motion is upward, we choose to point the unit vector $\underline{j}$ upward and so the weight of the projectile may be expressed as $\underline{W} = -mg\underline{j}$. Ignoring air resistance, the weight is the only force on the projectile, so

$$m\underline{a} = -mg\underline{j}. \qquad (1.5)$$

Note the minus sign which is a result of the choice of direction for $\underline{j}$. After dividing through by m, we obtain the vector equation for the acceleration due to gravity: $\underline{a} = -g\underline{j}$.

Recall that $\underline{a} = \dfrac{d\underline{v}}{dt}$ so, $\dfrac{d\underline{v}}{dt} = -g\underline{j}$.

Integrating this gives

$$\underline{v}(t) = -gt\underline{j} + \underline{c}.$$

Integrating again gives

$$\underline{r}(t) = -\frac{1}{2}gt^2\underline{j} + t\underline{c} + \underline{d} \qquad (1.6)$$

Since $\underline{r}(0) = \underline{0}$, $\underline{d} = \underline{0}$.

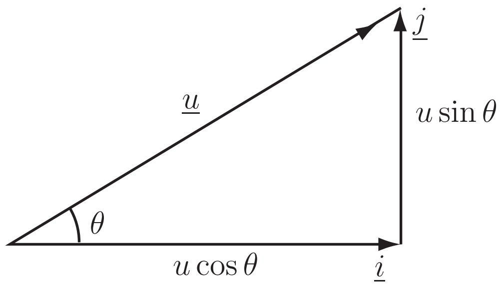


Figure 6: Components of launch velocity

The initial velocity may be expressed in vector form. Recall from HELM 9 that the component of a vector along a specific direction is given by the dot product of the vector with the unit vector in the direction of interest. The dot product involves the cosine of the angle between the vectors.

$$\underline{v}(0) = \underline{c} = u\cos\theta\underline{i} + u\sin\theta\underline{j} \quad \text{(from Figure 6).}$$

Hence

$$\underline{v}(t) = u\cos\theta\underline{i} + (u\sin\theta - gt)\underline{j} \qquad (1.7)$$

and

$$\underline{r}(t) = -\frac{1}{2}gt^2\underline{j} + t\underline{c} + \underline{d} = ut\cos\theta\underline{i} + \left(ut\sin\theta - \frac{1}{2}gt^2\right)\underline{j}. \qquad (1.8)$$

The vertical component of the position will be zero at the launch and when the projectile hits the ground. This will be true when

$$ut\sin\theta - \frac{1}{2}gt^2 = 0 \quad \text{so} \quad t(u\sin\theta - \frac{1}{2}gt) = 0.$$

This gives $t = 0$, as it should, or

$$t = \frac{2u\sin\theta}{g}.$$

At this value of t the horizontal position coordinate $ut\cos\theta$ will give the horizontal range, R, so

$$R = u\left(\frac{2u\sin\theta}{g}\right)\cos\theta = \frac{2u^2\sin\theta\cos\theta}{g},$$

or, since $\sin(2\theta) = 2\sin\theta\cos\theta$,

$$R = \frac{u^2 \sin 2\theta}{g}. \tag{1.9}$$

From this result it is possible to deduce that the maximum range, R_{max}, of a projectile measured at the same vertical level as its launch occurs when $\sin(2\theta)$ has its maximum value, which is 1, corresponding to $\theta = 45°$.

So, the maximum range is given by

$$R_{\max} = u^2/g. \tag{1.10}$$

From (1.8), the height (y coordinate) at any time is given by

$$y(t) = ut\sin\theta - \frac{1}{2}gt^2.$$

To find the maximum height, we can find the value of t at which $\dot{y}(t) = 0$. Once we have this value of t we can substitute in the expression for y to find the corresponding value of y. Note that the condition $\dot{y}(t) = 0$ at maximum height is the same as asserting that the vertical component of velocity must be zero at the maximum height. Hence, by differentiating $y(t)$ above, or from (1.7), it is required that

$u\sin\theta - gt = 0$, which gives $t = \dfrac{u\sin\theta}{g}$.

The height for any t is given by $y(t) = ut\sin\theta - \frac{1}{2}gt^2$. After substituting $t = \dfrac{u\sin\theta}{g}$ this becomes

$$y(t) = \frac{u^2\sin^2\theta}{g} - \frac{u^2\sin^2\theta}{2g} = \frac{u^2\sin^2\theta}{2g}.$$

Note that the time at which the projectile reaches its maximum height is exactly half the total time of flight ($t = \dfrac{2u\sin\theta}{g}$). In the absence of air resistance, the trajectory of the object will be a parabola with maximum height at its vertex which will occur halfway between the launch and the landing i.e. halfway through its flight. The horizontal coordinate of this point will be

$$\frac{R_{\max}}{2} = \frac{u^2\sin 2\theta}{2g}.$$

So the coordinates of the maximum height are

$$\left(\frac{u^2\sin 2\theta}{2g},\ \frac{u^2\sin^2\theta}{2g}\right). \tag{1.11}$$

If the trajectory corresponds to maximum range, i.e. $\theta = 45°$ (which means that $\sin^2\theta = \frac{1}{2}$), then the maximum height is $\dfrac{u^2}{4g}$ at a horizontal distance of $\dfrac{u^2}{2g}$ and the maximum range is $\dfrac{u^2}{g}$. Although several of these results are useful, particularly (1.10) for maximum range, and are worth committing to memory, it is more important to remember the method for deriving them from first principles.

Example 3

During a particular downhill run a skier encounters a short but sharp rise that causes the skier to leave the ground at 25 m s^{-1} at an angle of 30° to the horizontal. The ground immediately beyond the rise is flat for 60 m. Beyond this the downhill slope continues. See Figure 7. Ignoring air resistance, will the skier land on the flat ground beyond the rise?

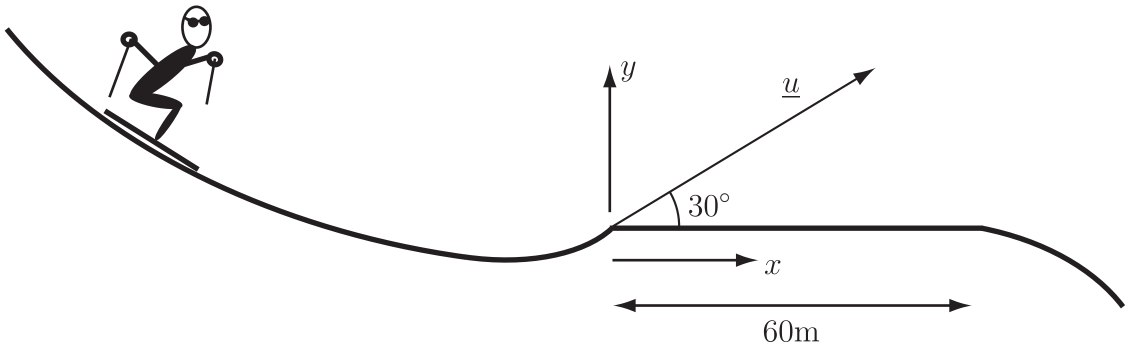


Figure 7: Coordinate system for skiing example

Solution

In this Example, from (1.9), $R = 625\sin(60)/g = 55.7$ m. So the skier will land on the flat part of the slope. The horizontal range achieved will be reduced by air resistance but increased if the skier is able to exploit aerodynamic lift from the skis during flight. In the absence of these effects, an effort to leave the ground at a slightly faster speed would be rewarded with the possibility of landing on the continuation of the downhill slope which may be an advantage in racing since it might reduce the interruption caused by the rise while picking up speed. A speed of 26.5 m s^{-1} at the same angle would mean that the skier lands beyond the flat ground at a range of 62 m from the rise.

A similar method to that used when considering a projectile launched from 'ground' level may be used if the projectile is launched at some height above the chosen origin of coordinates. If air resistance is ignored, the governing vector acceleration is still

$$\underline{a}(t) = -g\underline{j}$$

or

$$\frac{d\underline{v}}{dt} = -g\underline{j}.$$

Integrating this

$$\underline{v}(t) = -gt\underline{j} + \underline{c}.$$

Integrating again

$$\underline{r}(t) = -\frac{1}{2}gt^2\underline{j} + t\underline{c} + \underline{d}.$$

This time, instead of being launched from $y = 0$, the projectile is launched from $y = H$. So $\underline{r}(0) = H\underline{j}$, and hence $\underline{d} = H\underline{j}$. As before, the initial velocity may be expressed in vector form.

$$\underline{v}(0) = \underline{c} = u\cos\theta\underline{i} + u\sin\theta\underline{j}.$$

Hence $\underline{v}(t) = u\cos\theta\underline{i} + (u\sin\theta - gt)\underline{j}$ which is the same as (1.8). But, now

$$\underline{r}(t) = ut\cos\theta\underline{i} + \left(ut\sin\theta - \frac{1}{2}gt^2 + H\right)\underline{j} \qquad (1.12)$$

which differs from (1.8) by the extra H in the $\underline{j}$ component. Note that this single vector equation for $r(t)$ may be expressed as two separate equations for $x(t)$ and $y(t)$:

$$x(t) = ut\cos\theta \qquad y(t) = ut\sin\theta - \frac{1}{2}gt^2 + H.$$

Example 4

A stone is thrown upwards at 45° from a height of 1.5 m above flat ground and lands on the ground at a distance of 30 m from the point of launch. Ignoring air resistance, calculate the speed at launch.

Solution

For the particular case of interest $\theta = 45°$ and the stone lands at a horizontal distance of 30 m from the point of launch. Using these values in the $x(t)$ component of (1.12) gives $30 = \dfrac{ut}{\sqrt{2}}$. So the time for which the stone is in the air is $30\sqrt{2}/u$. Substitution of this time, at which $y = 0$, into the $y(t)$ component of (1.12) gives

$$0 = 30 - g\left(\frac{30}{u}\right)^2 + 1.5 \quad \text{or} \quad u = 30\sqrt{\frac{g}{31.5}} = 16.739 \ \text{ m s}^{-1}.$$

The speed of release is about 17 m s^{-1}.

Choosing trajectories

So far most of the projectiles we have considered have been launched without any particular control or target. However there are many instances in sport and recreation where there are clearly defined targets for the projectile motion and the trajectory is controlled through the speed and angle of launch. As we shall discover by considering several examples, it is possible to choose more than one path to achieve a given target. First we model a case in which the choice of angle is important.

Consider a projectile launched at an angle θ to the horizontal. If we take the origin of coordinates at the launch point, then, according to (1.7), the x- and y-coordinates of the projectile at time t are

$$x(t) = ut\cos\theta \qquad y(t) = ut\sin\theta - \frac{1}{2}gt^2.$$

Although the path is characterised parametrically in terms of t by these expressions, if we are given y or x or both instead of t, then it is useful to be able to express y in terms of x. We shall eliminate t, by substituting $t = \dfrac{x}{u\cos\theta}$ (which can be deduced from the first equation) in the second equation to give

$$y = x\tan\theta - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2.$$

By using $\dfrac{1}{\cos^2\theta} = \sec^2\theta = 1 + \tan^2\theta$, the equation for y may be written in the form:

$$y = x\tan\theta - \frac{1}{2}g\left(\frac{x}{u}\right)^2(1+\tan^2\theta). \qquad (1.13)$$

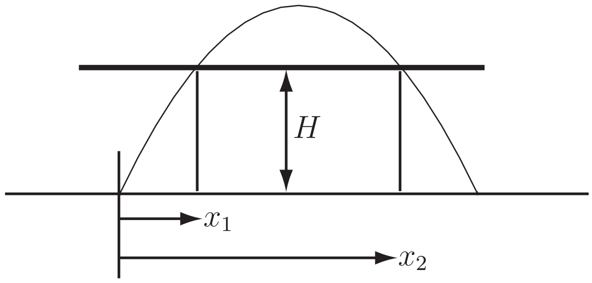


Figure 8: A projectile achieves a given height at two different ranges

For given values of u, θ and y, Equation (1.13) represents a quadratic in x. For a given speed and angle of launch, a given height is achieved at two different values of x. This is a consequence of the parabolic form of the trajectory (Figure 8). For given values of u, y and x, i.e. given a launch velocity and a target location, Equation (1.13) becomes a quadratic in $\tan\theta$ i.e.

$$\frac{g}{2}\left(\frac{x}{u}\right)^2\tan^2\theta - x\tan\theta + \frac{g}{2}\left(\frac{x}{u}\right)^2 + y = 0. \qquad (1.14)$$

Recall the condition $b^2 > 4ac$ for the quadratic $at^2 + bt + c = 0$ to have real roots. As long as

$$x^2 > 2g\left(\frac{x}{u}\right)^2\left(\frac{g}{2}\left(\frac{x}{u}\right)^2 + y\right) \qquad (1.15)$$

the quadratic will have two real roots.

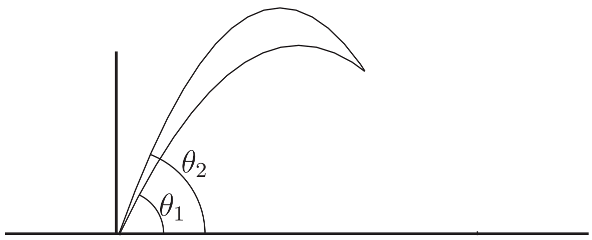


Figure 9: For a given speed and target location two different angles will attain the target

This means that two different choices of angle of launch will cause the projectile to pass through given coordinates (x, y); this is illustrated in Figure 9. If the projectile is launched from $y = H$ in the chosen coordinate system, then the $x(t)$ part of Equation (1.13) leads to the substitution $t = \dfrac{x}{u\cos\theta}$ as before, but the equation for y becomes

$$y = x\tan\theta - \frac{1}{2}g\left(\frac{g}{x}\right)^2\left(1+\tan^2\theta\right) + H \qquad (1.16)$$

which differs from (1.13) by the addition of H to the right-hand side.

Next we will look at two Examples of projectiles with chosen trajectories. In the first Example the influence of initial speed on the trajectory is important; in the second the influence of angle is important.

Example 5
(choosing speed)

As a result of many years of practice, a university teacher, is skilled at throwing screwed up sheets of paper, containing unsatisfactory attempts at setting examination questions, into a cylindrical waste paper bin. She throws at an angle of 20° above the horizontal and from 1.5 m above the floor. More often than not, the paper balls land in the bin which is 0.2 m high and has a radius of 0.15 m. The bin is placed so that its nearest edge is 3.0 m away (in a horizontal direction) from the point of launch. Model the paper ball as a projectile. Ignore air resistance and calculate the speed of throw that will result in the paper ball entering the bin at the centre of its open end.

Solution

We can choose the origin at the point of launch, with x- and y-axes as before (see Figure 10). In this case, we need to use the position vector (Equation (1.11)) and find the condition on the speed for the throw to be on target. In particular, we are given that $\theta = 20°$ and the location of the bin and need to determine the speed of throw necessary for the paper projectile to pass through the centre of the open end of the bin. The centre of the open end of the bin has the coordinates $(3.15, -1.3)$ with respect to the chosen origin. Note the negative value of the vertical coordinate since the top of the bin is located 1.3 m below the chosen origin.

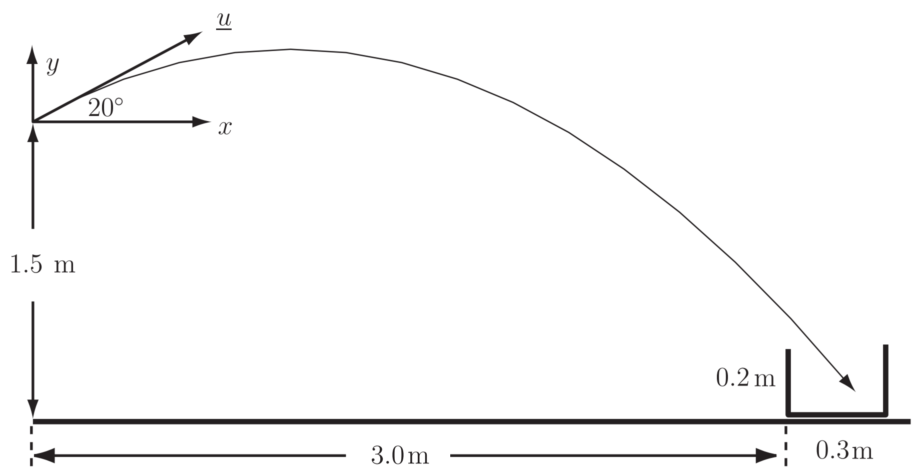


Figure 10: Path of screwed-up-paper projectiles

At the centre of the bin, using Equation (1.11) and the horizontal position coordinate, we have

$$ut\cos 20 = 3.15. \quad \text{so,} \quad t = \frac{3.15}{u\cos 20}.$$

Also, from (1.16) and the vertical coordinate

$$ut\sin 20 - \frac{1}{2}gt^2 = -1.3.$$

Solution (contd.)

Using $g = 9.81$ m s^{-2}, and the expression for t, gives

$$3.15\tan(20) - \frac{1}{2}9.81\left((3.15)^2/u^2\right)\sec^2 20 = -1.3,$$

which means that $u = 3.15\sec(20)\sqrt{\dfrac{0.5(9.81)}{1.3 + 3.15\tan(20)}}$.

Hence $u =$ 4.746 m s^{-1} i.e. the academic throws the screwed up paper at about 4.75 m s^{-1}.

Clearly the motion of screwed up pieces of paper will depend to a significant extent on air resistance. We shall consider how to model resisted motion later (Section 34.3).

Task

According to the model developed above, for what range of throwing speeds will the academic be successful in getting the paper ball into the basket?

Your solution

Answer

Assume that the time of flight of the screwed up paper balls and the angle of throw do not change. The permitted variation in throwing speed is determined by the horizontal distance $ut\cos 20$. The screwed up paper ball will not find the bin if this product is less than 3 m or more than 3.3 m. Using $t = (3.15/4.75)\cos 20$ with $ut\cos 20$ ranging from 3 to 3.3 gives $4.524 \leq u \leq 4.976$.

Example 6
(choosing angle)

In the game of Tiddly-winks, small plastic discs or counters ('tiddly-winks' or 'winks') are caused to spring into the air by exerting sharp downward pressure at their edges with another (usually larger) disc called a squidger. By changing the pressure and overlap of the larger disc it is possible to control the velocity at launch of each wink. The object of the game is to 'pot' all of the winks into a cup or cylindrical receptacle before your opponent does. One important skill when 'potting' is to be able to clear the edge of the collecting cup with the winks. For a given speed at launch, an experienced or successful player will know how the path changes with angle. Suppose that air resistance can be ignored and that a wink may be modelled as a point object and its spin may be ignored.

Given that the nearest edge of the cup is 0.05 m high,

(a) Calculate (i) the speed of launch such that the maximum height on the maximum range trajectory is 0.05 m and (ii) the associated maximum range.

(b) Given a launch speed that is 0.1 m s^{-1} faster than that calculated in (a) find the angle of launch that is likely to be successful for potting the wink when the centre of the cup is 0.1 m from the point of launch.

Solution

(a) The expression for maximum height (Equation (1.11)) may be used to calculate a corresponding speed of launch.

Hence $\dfrac{u^2}{4g} = 0.05$, or $u = \sqrt{0.2g} = 1.4$ so the required speed of launch is 1.4 m s^{-1}.

For this trajectory the maximum height is reached at a horizontal distance of $\dfrac{u^2}{2g} = 0.1$ m from launch and the maximum range is 0.2 m (see Figure 11).

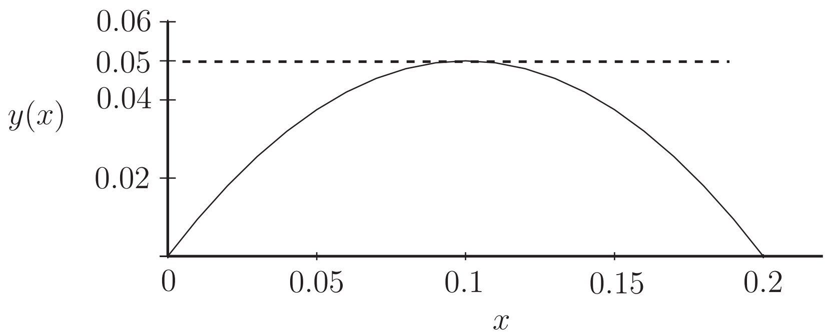


Figure 11: Maximum range trajectory of wink achieving a maximum height of 0.05 m

(b) Although the wink, travelling on the trajectory shown in Figure 11 would reach the edge of the cup i.e. a height of 0.05 m at 0.1 m range, it would not necessarily enter the cup. The finite size of the wink might mean that it hits the edge of the cup and falls back. The wink is more likely to enter the cup if it is descending when it encounters the cup.

Solution (contd.)

In this case the launch speed and target coordinates are specified so Equation (1.16) can be brought into play. If we take $x = 0.1$ m, $u = 1.5$ m s^{-1}, and $y\ (= H) = 0.05$ m, then it turns out that condition (1.15) is satisfied ($x^2 = 0.01$ and $2g\left(\frac{x}{u}\right)^2\left(\frac{g}{2}\left(\frac{x}{u}\right)^2 + y\right) = 0.006$) and there are two values for θ, which are 41.5° and 71.8°.

The corresponding trajectories $y1(x)$ and $y2(x)$ are shown in Figure 12. The smaller angle results in the shallower trajectory (solid line). The larger angle produces the required result (dotted line) that the wink is descending at $x = 0.1$ m and hence is more likely to enter the cup. This assumes that the cup is at least 2 cm wide.

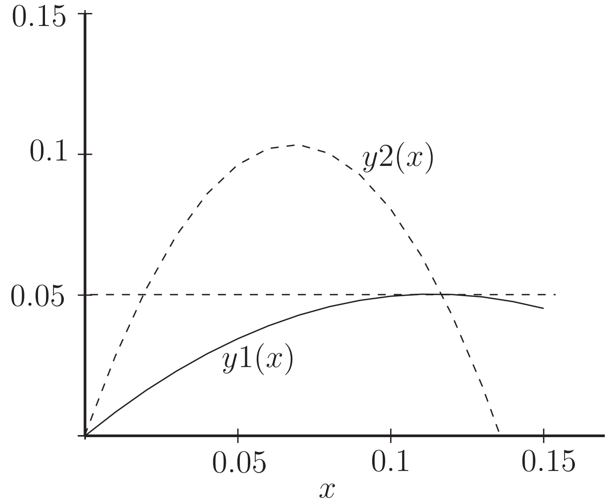


Figure 12

Two trajectories corresponding to $x = 0.1, y = 0.05, u = 1.5$ m s^{-1} that pass through (0.1, 0.05)

Example 7
(choosing the angle again)

An engineering student happens to be a fine shot-putter. At a tutorial on projectiles he argues that, because he throws from a height of about 2 m, he needs to launch the shot at an angle other than 45° to get the greatest range. He claims that when launching at 45° the furthest he can put the shot is to a horizontal distance of 17 m from the launch.

(a) Calculate the speed at which he releases the shot at 45° ignoring air resistance.

(b) Write down an equation for the trajectory of the shot, assuming that the shot is released always at the maximum speed calculated in (a). Set the vertical coordinate to zero and substitute the constant L for the maximum range at the height of launch to obtain a quadratic for the horizontal range R.

(c) Hence, by differentiating the resulting equation with respect to R, calculate the optimum angle of launch and the maximum range.

Solution

(a) For the particular case of interest $\theta = 45^\circ$ and the shot lands at a horizontal distance of 17 m from the point of launch. Using these values, (1.12) gives

$$17 = ut\cos(45^\circ) = \frac{ut}{\sqrt{2}}$$

So the time for which the shot is in the air, i.e. before it lands, is $17\sqrt{2}/u$. Substitution of this time (at which $y = 0$) into the y part of (1.12) gives

$$0 = 17 - g\left(\frac{17}{u}\right)^2 + 2 \quad \text{or} \quad u = 17\sqrt{\frac{g}{19}} = 12.2 \ \text{ m s}^{-1}.$$

The speed of release is about 12 m s^{-1}. According to the shot-putter this is more or less his maximum speed of release.

(b) The general equation for the trajectory of the shot is (1.16)

$$y = x\tan\theta - \frac{1}{2}g\left(\frac{x}{u}\right)^2(1+\tan^2\theta) + H.$$

Given that the maximum speed of release and optimum angle of launch are employed, the shot should land at the maximum range, R. From the general Equation (1.16), with $x = R$ and $y = 0$, we have

$$0 = R\tan\theta - \frac{1}{2}g\left(\frac{R}{u}\right)^2(1+\tan^2\theta) + H.$$

Solution (contd.)

Since $\dfrac{u^2}{g}$ does not depend on either R or θ, we replace it by a constant L, and rearrange the equation into the usual form for a quadratic in R:

$$0 = -\frac{R^2}{2L}(1 + \tan^2\theta) + R\tan\theta + H.$$

(c) The optimum angle of launch is found by obtaining an expression for R and setting $\dfrac{dR}{\mathrm{d}\theta}$ equal to zero. As you can imagine, the expression for R resulting from solving this quadratic is rather complicated and nasty to differentiate. An alternative approach is called **implicit differentiation**. (See HELM 11.7). We work through the equation as it stands differentiating term by term with respect to θ and making use of the relationship

$$\frac{df(R)}{d\theta} = \frac{df(R)}{dR} \times \frac{dR}{d\theta}.$$

(For example, $\dfrac{d(R^2)}{\mathrm{d}\theta} = 2R\dfrac{dR}{d\theta}$). Hence implicit differentiation gives

$$0 = -\frac{1}{2L}\left(2R\frac{dR}{d\theta}\right)(1 + \tan^2\theta) - \frac{R^2}{2L}(2\tan\theta\sec^2\theta) + \frac{dR}{d\theta}\tan\theta + R\sec^2\theta.$$

At the maximum range, $\dfrac{dR}{\mathrm{d}\theta} = 0$:

$$0 = -\frac{R^2}{2L}(2\tan\theta\sec^2\theta) + R\sec^2\theta \quad \text{so} \quad R\sec^2\theta\left(-\frac{R}{L}\tan\theta + 1\right) = 0.$$

Since $\sec^2\theta$ cannot be zero and the option of $R = 0$ is not very interesting, it is possible to conclude that the relationship between the optimum angle of launch and the maximum range is given by

$$\tan\theta = \frac{L}{R}.$$

This result may be substituted back into the quadratic for R to give

$$0 = -\frac{R^2}{2L}\left(1 + \frac{L^2}{R^2}\right) + L + H \quad \text{or} \quad 0 = -\frac{R^2}{2L} - \frac{L}{2} + L + H.$$

Multiplying throughout by $2L$ gives

$$R^2 = L^2 + LH \quad \text{i.e.} \quad R = \sqrt{(L^2 + 2LH)}.$$

A consequence of this result is that $R > L$. Bearing in mind that $L = u^2/g$ is the maximum range at the height of the launch (or for a launch at ground level), this means that the maximum range from the elevated launch to ground level is greater than the maximum range in the plane at the height of the launch. Substituting the result for R in the result for $\tan\theta$ gives

$$\tan\theta = \frac{L}{\sqrt{L^2 + 2HL}} = \frac{1}{\sqrt{1 + \dfrac{2H}{L}}}.$$

Solution (contd.)

For $H > 0$, this implies that $\tan\theta < 1$, which in turn implies an optimum angle of launch $< 45°$ and that the shot putter's assertion is justified. When a projectile is launched at some angle from some point above ground level to land on the ground then the optimum angle of launch is less than 45°. Specifically, if $u = 12.2$ m s^{-1} and $H = 2$ m, then $L = 15.2$ m, $R = 17.06$ m and $\tan\theta = 0.9 \quad (\theta = 41.7°)$.

The 45° launch trajectory and optimum angle launch trajectory are shown in Figure 13 together with the maximum range trajectory for a ground level launch. A close up of the ends of the first two trajectories is shown in Figure 14. The shot-putter can increase the length of his putt only by a few centimetres if he putts at the optimum angle of launch rather than 45°. However these could be a vital few centimetres in a tight competition!

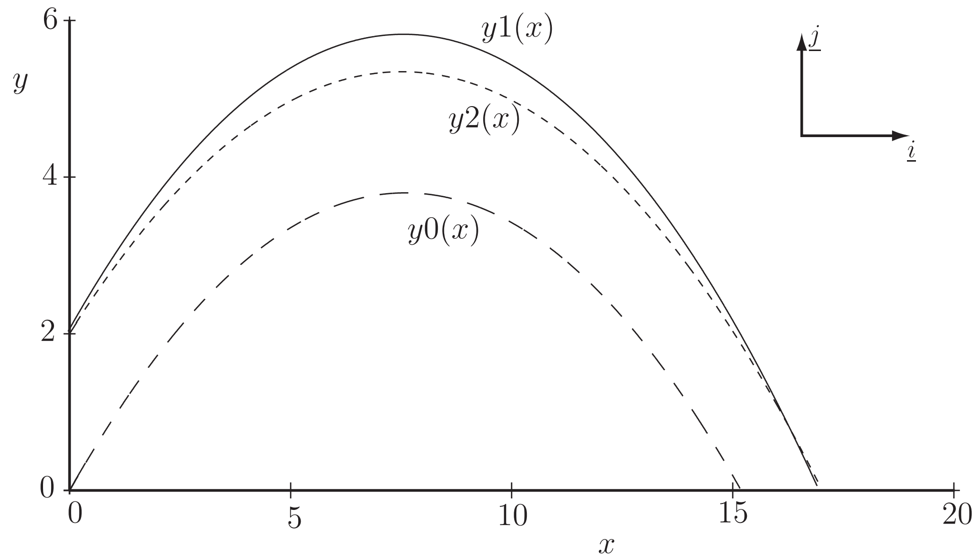


Figure 13: 45° trajectory (solid line) and optimum angle trajectories for a shot-put

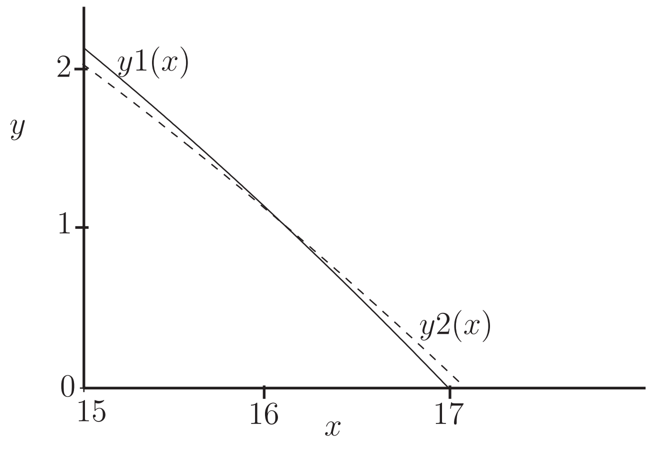


Figure 14: Close up of ends of trajectories in Figure 13

Task

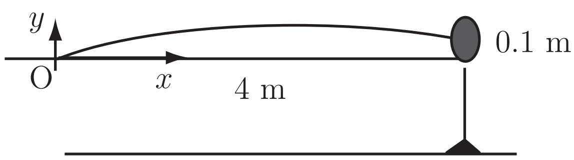
A fairground stall known as a 'coconut shy' consists of an array of coconuts placed on stands. The objective is to win a coconut by knocking it off its stand with a wooden ball. A local youth has learned that if he throws a wooden ball as fast as he can at 10° above the horizontal he is able to hit the nearest coconut more or less dead centre and knock it down almost every time. The nearest coconut stand is located 4 m from the throwing position with its top at the same height as the balls are thrown. The coconuts are 0.1 m long.

(a) Calculate how fast the youth is able to throw if air resistance is ignored:

Your solution

Answer

Choose an origin of coordinates at the point of launch of the wooden balls, with the y-coordinate in the vertical upwards direction and the x-coordinate along the horizontal towards the coconut.



Possible trajectory of ball at fairground 'coconut shy' stall

If air resistance is ignored, the trajectory of the balls may be modelled by the Equation (1.13)

$$y = x\tan\theta - \frac{1}{2}g\left(\frac{x}{u}\right)^2(1+\tan^2\theta).$$

If the balls hit the coconut dead centre, then their trajectory must pass through the coordinates (4, 0.05) (see diagram above). Hence

$$y = x\tan\theta - \frac{1}{2}g\left(\frac{x}{u}\right)^2(1+\tan^2\theta)$$

so $\left(\dfrac{4}{u}\right)^2 = \dfrac{2}{g}\dfrac{(4\tan 10 - 0.05)}{1+\tan^2 10}$ or $u = 4\sqrt{\dfrac{g}{2}\dfrac{1+\tan^2 10}{(4\tan 10 - 0.05)}} = 11.1.$

So the youth is able to throw at 11.1 m s^{-1}.

(b) Calculate how much further the operator of the fairground stall should move the cocunuts from the throwing line to prevent the youth hitting the coconut so easily:

Your solution

Answer

The youth will fail if the nearest coconut is moved sufficiently far away so that the trajectory considered in part (a) passes beneath the bottom of the coconut on top of the stand. If x is the horizontal range corresponding to a y-coordinate of 0, then, using the expression in Equation (1.9) on page 13:

$$x = \frac{u^2 \sin 2\theta}{g}. \quad \text{In this case,} \qquad x = \frac{(11.1)^2 \sin 20}{9.81} = 4.296.$$

So the nearest coconut stand should be moved another 0.296 m from the throwing line. This has assumed that the youth will favour as 'flat' a trajectory as possible. The youth could choose to throw at a greater angle to increase the range. For example throwing at an angle of 45° would result in a range of 12.6 m. However the steeper the angle of launch, the greater will be the angle to the horizontal at which the ball arrives at the coconut. A large angle would not be as efficient as a small one for dislodging it.

Task

Basketball players are able to gain three points for long-range shooting. The shot must be made from outside a certain radius from the basket. A skilled player makes a jump shot rather than standing on the ground to shoot. He leaps so that he is able to project the ball at a slower, i.e. more controllable, speed and from the same height as the basket, which is 3 m above the ground. Assume that the ball would be released from a height of 2 m when the player is standing on the ground and that air resistance can be ignored.

(a) Calculate the speed of release during a jump shot made at a horizontal distance of 12 m from the basket at maximum range for that speed of release:

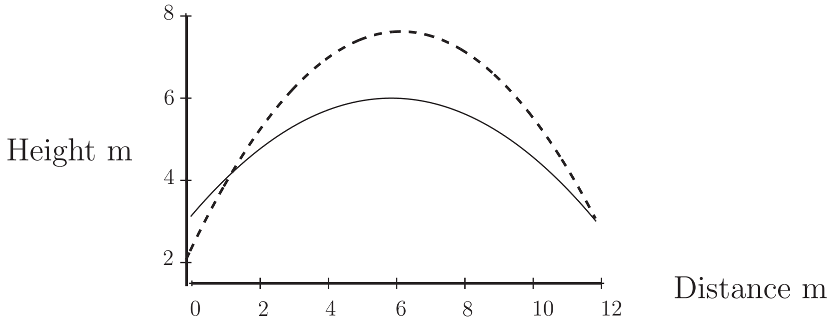
Your solution

Answer
If the maximum range is 12 m, then, since the jump shot is made at the same height as the basket, $u^2/g = 12$, i.e. $u = 10.85$, so the speed of release is 10.85 m s^{-1}.

(b) Calculate the preferred angle of launch that would hit the basket if the shot were to be made when the player is standing at the same point and shoots at 12 m s^{-1}:

Your solution

Answer
Use may be made of Equation (1.15), in the form of a quadratic for $\tan(\theta)$, where θ is the launch angle, with $y = 3, x = 12, H = 2$. The largest root corresponds to the preferred form of trajectory for passing into the basket (see diagram below) and gives $\tan(\theta) = 1.764$ or $\theta = 60.5°$.



Trajectories for a jump-shot (solid line) and a shot-from-the-floor (dashed line)

4. Energy and projectile motion

In this Section we demonstrate that, in the absence of air resistance, energy is conserved during the flight of a projectile. Consider first the launch of an object vertically with speed u. In the absence of air resistance, the height reached by the object is given by the result obtained in Equation (1.9) on page 13 i.e. $\dfrac{u^2 \sin^2 \theta}{2g}$ with $\theta = 90°$, so the height is $\dfrac{u^2}{2g}$. This result was obtained in Equation (1.10) by considering time of flight. However it can be obtained also by considering the energy. If the highest point reached by the object is h above the point of launch, then with respect to the level of launch, the **potential energy** of the object at the highest point is mgh. Since the vertical velocity is zero at this point then mgh represents the **total energy** also. At the launch, the potential energy is zero and the total energy is given by the **kinetic energy** $\dfrac{1}{2}mu^2$. Hence, according to the conservation of energy

$$\frac{1}{2}mu^2 = mgh \quad \text{or} \quad h = \frac{u^2}{2g}$$

as required. Now we will repeat this analysis for the more general case of an angled launch and for any point (x, y) along the trajectory. Let us use the general results for the position vector and velocity vector expressed in Equations (1.4) and (1.5) on page 12. In the absence of air resistance, the horizontal component of the projectile velocity (v) is constant. If the height of the launch is taken as the reference level, then the potential energy at any time t and height y is given by

$$mgy = mg\left(ut\sin\theta - \frac{1}{2}gt^2\right).$$

The kinetic energy is given by

$$\begin{aligned} \frac{1}{2}m|v|^2 &= \frac{1}{2}m\left[(u\sin\theta - gt)^2 + u^2\cos^2\theta\right] \\ &= \frac{1}{2}m\left(u^2 - 2gtu\sin\theta + g^2t\right) = \frac{1}{2}mu^2 - mgx. \end{aligned}$$

So

$$\frac{1}{2}m|v|^2 + mgx = \frac{1}{2}mu^2.$$

But the initial kinetic energy, which is the initial total energy also, is given by $\dfrac{1}{2}mu^2$. Consequently we have shown that energy is conserved along a projectile trajectory.

5. Projectiles on inclined planes

The forces acting on an object resting on a sloping surface are its weight $\underline{W}$, the normal reaction $\underline{N}$, and the frictional force $\underline{R}$. Since all of the forces act in a vertical plane then they can be described with just two axes (indicated by the unit vectors $\underline{j}$ and $\underline{k}$ in Figure 15). If there is negligible friction ($\underline{R} = \underline{0}$) then, of course, the object will slide down the plane. If we apply Newton's second law to motion in the frictionless inclined plane, then the only remaining force to be considered is $\underline{W}$, since $\underline{N}$ is normal to the plane. Resolving in the $\underline{j}$-direction gives the only force as

$$-|\underline{W}|\cos(90-\alpha)\underline{j} = -|\underline{W}|\sin\alpha\underline{j}.$$

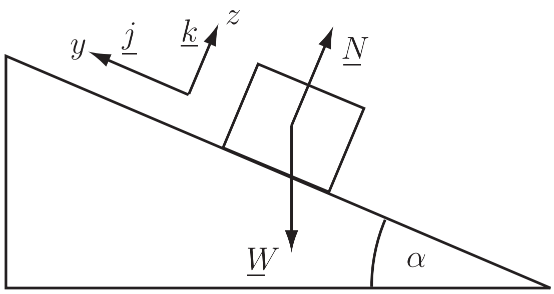


Figure 15: Mass on an inclined plane

This will be the only in-plane force on an object projected across the inclined plane, moving so that it is always in contact with the plane, and projected at some angle θ above the horizontal in the plane (as in Figure 16 for Example 8). Newton's second law for such an object may be written

$$m\underline{a} = -mg\sin\alpha\underline{j} \quad \text{or} \quad \underline{a} = -g\sin\alpha\underline{j}.$$

The resulting acceleration vector differs from that considered in HELM 34.2 Subsection 2 only by the constant factor $\sin\alpha$. In other words, the ball will move on the inclined plane as a projectile under reduced gravity (since $g\sin\alpha < g$).

Suppose that the object has an intial velocity $\underline{u}$. This is given in terms of the chosen coordinates by

$$u\cos\theta\underline{i} + u\sin\theta\underline{j}$$

which is the same as that considered earlier in Section 34.1. So it is possible to use the result for the range obtained in Section 34.1 Equation (1.6), after remembering to replace g by $g\sin\alpha$. Equations (1.9) to (1.11) may be applied as long as g is replaced by $g\sin\alpha$.

Example 8

In a game a small disc is projected from one corner across a smooth board inclined at an angle α to the horizontal. The disc moves so that it is always in contact with the board. The speed and angle of launch can be varied and the object of the game is to collect the disc in a shallow cup situated in the plane at a horizontal distance d from the point of launch. Calculate the speed of launch at an angle of 45° in the plane of the board that will ensure that the disc lands in the cup.

Solution

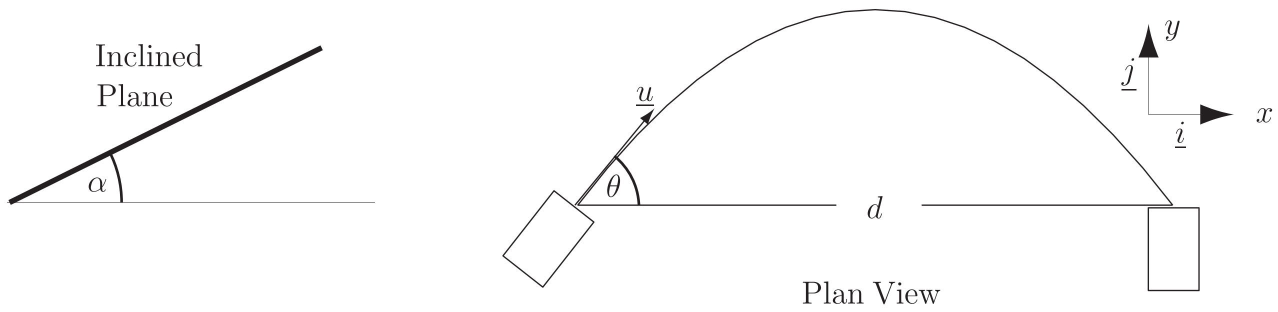


Figure 16

Sketches of inclined plane and the desired trajectory in the plane for Example 8

Consider a point along the disc's path portrayed from the side and looking down on the plane of the board in Figure 16. We choose coordinates and unit vectors as shown in these figures, so that y is up the plane and x along it, while z is normal to the plane. Equation (1.11) will apply as long as g is replaced by $g \sin \alpha$. So the range is given by $R = \dfrac{u^2 \sin 2\theta}{g \sin \alpha}$.

When $R = d$ and $\theta = 45°$, this gives $d = \dfrac{u^2}{g \sin \alpha}$ or $u = \sqrt{gd \sin \alpha}$

Note that d represents the maximum 'horizontal' , i.e. in-plane, range from the point of launch for this launch speed.

Task

Suppose that, in the game featured in the last example, there is another cup in the plane with the centre of its open end at coordinates $(3d/5, d/5)$ with respect to the point of launch, and that a successful 'pot' in the cup will gain more points. What angle of launch will ensure that the disc will enter this second cup if the magnitude of the launch velocity is $u = \sqrt{gd\sin\alpha}$?

Your solution

Answer

Equation (1.11) can be used with g replaced by $g\sin\alpha$ i.e.

$$y = x\tan\theta - \frac{1}{2}g\sin\alpha\left(\frac{x}{u}\right)^2(1+\tan^2\theta).$$

With $y = d/5,\ x = 3d/5, u = \sqrt{gd\sin\alpha}$, it is possible to obtain a quadratic equation for $\tan\theta$:

$$9\tan^2\theta - 30\tan\theta + 19 = 0.$$

Hence the required launch angle is approximately 68° from the 'horizontal' in the plane.

Task

Skateboarders have built jumps consisting of short ramps angled at about 30° from the horizontal. Assume that the speed on leaving the ramp is 10 m s^{-1} and ignore air resistance.

(a) Write down appropriate position, velocity and acceleration vectors:

Your solution

Answer
Ignoring air resistance, the relevant vectors are

$$\underline{r}(t) = \begin{bmatrix} (u\cos\theta)t \\ \\ u\sin\theta - \dfrac{1}{2}gt^2 \end{bmatrix}, \ \underline{v}(t) = \begin{bmatrix} u\cos\theta \\ \\ -gt \end{bmatrix}, \qquad \underline{a}(t) = \begin{bmatrix} 0 \\ \\ -g \end{bmatrix}.$$

In the skateboarders case, $u = 10$ and $\theta = 30°$, so the vectors may be written

$$\underline{r}(t) = \begin{bmatrix} 8.67t \\ \\ 5 - \dfrac{1}{2}gt^2 \end{bmatrix}, \ \underline{v}(t) = \begin{bmatrix} 8.67 \\ \\ -gt \end{bmatrix}, \qquad \underline{a}(t) = \begin{bmatrix} 0 \\ \\ -g \end{bmatrix}.$$

(b) Predict the maximum length of jump possible at the level of the ramp exits:

Your solution

Answer
The horizontal range x m at the level of exit from the ramp is given by

$$x = \frac{u^2 \sin 2\theta}{g} = \frac{100 \sin 60}{9.81} = 8.828.$$

(c) Predict the maximum height of jump possible measured from the ramp exits:

Your solution

Answer
From Section 34.1 Equation (1.11), the maximum height y_m m measured from the ramp exit is given by

$$y_m = \frac{u_v^2}{2g} = \frac{u^2 \sin^2\theta}{2g} = \frac{25}{2 \times 9.81} = 1.274.$$

(d) Comment on the choice of slope for the ramp:

Your solution

Answer

For a fixed value of u, the maximum range $\dfrac{u^2 \sin 2\theta}{g}$ is given when $\theta = 45°$. On the other hand, the maximum height $\dfrac{u^2 \sin^2 \theta}{2g}$ is given when $\theta = 90°$. The latter would require vertical ramps and these are not very practicable!

Forces in More than One Dimension

34.2

Introduction

This Section looks at forces on objects resting or moving on inclined planes and forces on objects moving along curved paths. The previous ideas are exploited in example calculations related to passenger sensations of forces on amusement rides.

Prerequisites

Before starting this Section you should ...

- be able to use vectors and to carry out scalar and vector products
- be able to use Newton's laws to describe and model the motion of particles
- be able to use coordinate geometry to study circles and parabolas
- be able to use calculus to differentiate and integrate polynomials

Learning Outcomes

On completion you should be able to ...

- compute frictional forces on static and moving objects on inclined planes and on objects moving at constant speed around bends
- calculate the forces experienced by passengers in vehicles moving along straight, curved and inclined tracks

1. Forces in two or three dimensions

Forces during circular motion

Consider a particle moving in a horizontal plane so that its position at any time t is given by

$$\underline{r} = r\cos\theta\underline{i} + r\sin\theta\underline{j}$$

where r is a constant and $\underline{i}$ and $\underline{j}$ are unit vectors at right-angles. The angle, θ, made by $\underline{r}$ with the horizontal is a function of time. We can consider four special values of θ and the associated values of $\underline{r}$. These are shown in the following table and in Figure 17.

θ	$\underline{r}$
0	$r\underline{i}$
$\pi/2$	$r\underline{j}$
π	$-r\underline{i}$
$3\pi/2$	$-r\underline{j}$

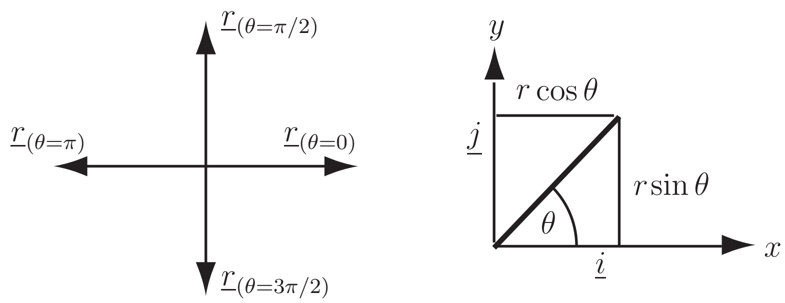


Figure 17

Note that $|\underline{r}| = r$ is a constant for all values of θ, so we must have motion in a circle of radius r. If we assume a constant angular velocity ω rad s^{-1} so that $\theta = \omega t$, then the velocity is

$$\frac{d\underline{r}}{dt} = \omega r(-\sin\omega t\underline{i} + \cos\omega t\underline{j}). \qquad (2.1)$$

Hence, taking the dot product,

$$\underline{r}.\frac{d\underline{r}}{dt} = \omega r^2(-\cos\omega t\sin\omega t + \sin\omega t\cos\omega t) = 0$$

which implies that $\dfrac{d\underline{r}}{dt}$ is always perpendicular to $\underline{r}$. Since $\dfrac{d\underline{r}}{dt}$ is the velocity vector $\underline{v}$, this means that the velocity vector is always tangential to the circle (see Figure 18). Note also that $|\underline{v}| = v = \omega r$, so $\omega = \dfrac{v}{r}$. Differentiating (2.1) again,

$$\frac{d^2\underline{r}}{dt^2} = \omega^2 r(-\cos\omega t\underline{i} - \sin\omega t\underline{j}) = -\omega^2\underline{r}. \qquad (2.2)$$

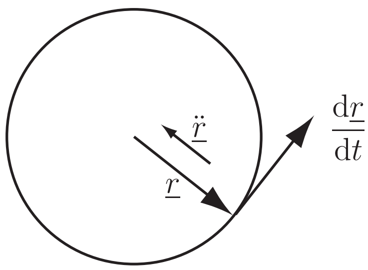


Figure 18: The velocity vector is tangential to motion in a circle

Equation (2.2) means that the second derivative, $\frac{d^2\underline{r}}{dt^2}$, which represents the acceleration $\underline{a}$, acts along the radius towards the centre of the circle and is perpendicular to $\frac{d\underline{r}}{dt}$.
The **magnitude** of the velocity (the speed) is constant and the acceleration, $\underline{a}$, is associated with the changing **direction** of the velocity. The force must act towards the centre of the circle to achieve this change in direction around the circle. Since $\underline{a}(t) = -\omega^2\underline{r}(t)$, where $\omega = \frac{v}{r}$, we see that the acceleration acts towards the centre of the circle and has a magnitude given by $a = \frac{v^2}{r}$. This is a special example of the fact that forces in the direction of motion cause changes in speed, while forces at right-angles to the direction of motion cause changes in direction.

When a particle is moving at constant speed around a circle on the end of a rope, then the force directed towards the centre is supplied by the tension in the rope. When a vehicle moves at constant speed around a circular bend in a road, then the force directed towards the centre of the bend is supplied by sideways friction of the tyres with the road. If the vehicle of mass m were to be pushed or dragged sideways by a steady force then it would be necessary to overcome the frictional force. This force depends on the normal reaction $\underline{R}$, which is equal and opposite to the weight of the vehicle (mg). The friction force is given by μmg where μ is the coefficient of friction and it must at least equal the required force towards the centre of the bend to avoid skidding. So, we must have

$$\mu mg \geq \frac{mv^2}{r}. \qquad (2.3)$$

Example 9

A car of mass 900 kg drives around a roundabout of radius 15 m at a constant speed of 10 m s^{-1}.

(a) Draw a vector diagram showing the forces on the car in the vertical and sideways directions.

(b) What is the magnitude of the force directed towards the centre of the bend?

(c) What is the friction force between the car and the road?

(d) What does this imply about the minimum value of the coefficient of friction?

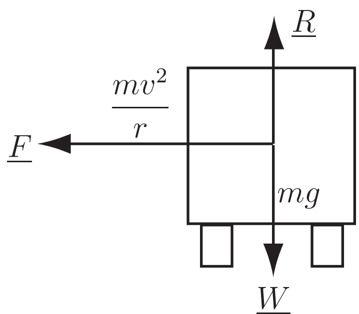


Figure 19: Forces on a vehicle negotiating a circular bend at constant speed

Solution

(a) See Figure 19. (In addition to the sideways friction involved in cornering, there will be a net force causing forward motion which is generated by the vehicle engine and exerted through friction between the tyres and the road.) The forces are shown as if they act at the centre of the vehicle, since the vehicle is being treated as a particle. Strictly speaking, the frictional forces on a road vehicle should be considered to act at the tyre/road contact and there will be differences between the forces at each wheel.

(b) The magnitude of the force is obtained by using

$$\frac{mv^2}{r} = \frac{900 \times 100}{15} = 6000.$$

So the magnitude of the force acting towards the centre of the roundabout is 6000 N.

(c) The sideways force provided by friction is $|\underline{F}| = \mu|\underline{R}|$. In this case $|\underline{R}| = mg$.

(d) Consequently we must have $\mu mg \geq 6000$, that is

$$\mu \geq \frac{6000}{900 \times 9.81} = 0.68.$$

Task

Suppose that the coefficient of friction between the car and the ground in dry conditions is 0.96.

(a) At what speed could the car drive around the roundabout without skidding?

Your solution

Answer

Equation (2.3) on page 36 states that for the vehicle to go round the bend just without skidding

$$\frac{mv^2}{r} = \mu mg, \text{ or } \frac{v^2}{r} = \mu g.$$

In this case, the maximum speed is required, so the relationship is best rearranged into the form $v = \sqrt{\mu g r}$. The values to be substituted are $\mu = 0.96$, $g = 9.81$ and $r = 15$, so

$$v = \sqrt{0.96 \times 9.81 \times 15} = 11.9.$$

So the car will skid if it drives round the roundabout at more than 11.9 m s^{-1} (nearly 27 mph).

(b) What would be the radius of roundabout that would enable a car to drive around it safely in dry conditions at 30 m s^{-1} (nearly 70 mph)?

Your solution

Answer

For this part of the question, the speed around the roundabout is known and the safe radius is to be found. Further rearrangement of the expression used in part (a) gives

$$r = \frac{v^2}{\mu g}.$$

Hence

$$r = \frac{30^2}{0.96 \times 9.81} = 95.6.$$

In reality drivers should not be exactly at the limits of the friction force while going round the roundabout. There should be some safety margin. So the roundabout should have a radius of at least 100 m to allow cars to drive round it at 30 m s^{-1}.

(c) What would be the safe radius of this roundabout when conditions are wet so that the coefficient of friction between the car and the road is reduced by a factor of 2?

Your solution

Answer

The form of the equation used in part (b) indicates that if the coefficient of friction is halved (to 0.48) by wet conditions, then the safe radius should be doubled.

When a cyclist or motorcyclist negotiates a circular bend at constant speed, the forces experienced at the points of contact between the tyres and the road are a frictional force towards the centre of the bend and the upward reaction to the combined weight of the cyclist and the cycle (see Figure 20). These forces can be combined into a resultant that acts along an angle to the vertical. Suppose that the combined mass is m and that the coefficient of friction between the tyres and road surface is μ.

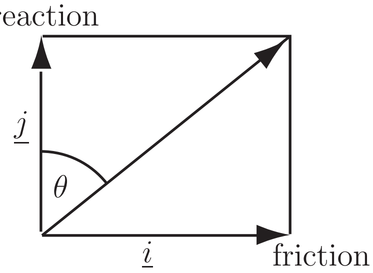


Figure 20
Forces on cycle tyres and the angle for cyclist comfort on a circular bend.
The net driving force is ignored.

The total force vector may be written

$$\underline{F} = mg(\mu\underline{i} + \underline{j})$$

and the angle θ is given by

$$\tan\theta = \frac{\mu mg}{mg} = \mu \qquad \text{so that} \qquad \theta = \tan^{-1}\mu.$$

Also, as argued previously (Equation (2.3)), we must have $\mu \geq \dfrac{v^2}{gr}$, which means that

$$\theta \geq \tan^{-1}\left(\frac{v^2}{gr}\right)$$

To be comfortable while riding, the cyclist likes to feel that the total force is vertical. So when negotiating the bend, the cyclist tilts towards the bend so that the resultant force acts along a 'new vertical'.

Example 10

(a) Calculate the angular velocity of the Earth in radians per second, assuming that the Earth rotates once about its axis in 24 hours.

(b) A synchronous communications satelite is launched into an orbit around the equator and appears to be stationary when viewed from the Earth. Calculate the radius of the satellite's orbit, given that $g = 9.81$ m s^{-2} and that the radius of the Earth is 6.378×10^6 metres.

Solution

(a) The angular velocity of the Earth is

$$\frac{2\pi}{24 \times 60 \times 60} = 7.272 \times 10^{-5} \text{ radians per second.}$$

(b) According to Newtonian theory of gravitation the attraction due to gravity at the Earth's surface, for a mass m, should be $\dfrac{GMm}{R^2}$, which is set equal to mg in elementary calculations. Thus we must have $g = \dfrac{GM}{R^2}$, so that the product GM equals

$$gR^2 = 9.81 \times (6.378)^2 \times 10^{12} = 3.991 \times 10^{14} \text{ m}^3/\text{s}^2.$$

For a circular satellite orbit of radius r, the gravitational force must equal mass times inward acceleration. For a mass m travelling at orbital speed v and with orbital angular velocity ω, the theory of circular orbits gives the result that the inward acceleration is

$$\frac{v^2/r}{\omega^2 r}.$$

The equation 'force equals mass times acceleration' thus gives:

$$\frac{GMm}{r^2} = m\omega^2 r.$$

We wish to ensure that the value of ω for the satellite orbit equals the value of ω for the Earth's rotation. The equation above gives the result: $\quad r^3 = \dfrac{GM}{\omega^2} = \dfrac{gR^2}{\omega^2}$

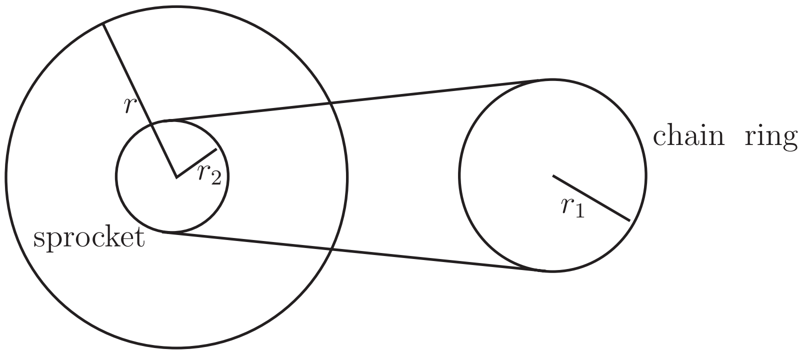
and the value of ω to be used is that which has already been calculated. This gives the result

$$r^3 = \frac{3.991 \times 10^{14}}{(7.272 \times 10^{-5})^2} = 7.547 \times 10^{22} \text{ m}^3$$

Taking the cube root gives $r = 4.23 \times 10^7$ metres. The radius of the satellite orbit is thus about 6.6 times the Earth's radius.

Task

The pedals on a bicycle drive the chain ring, which moves the chain. The chain passes around the sprocket (gear wheel) attached to the rear wheel and hence the rear wheel and the bicycle are driven (see diagram). There are cog teeth of equal width (d) cut into both the chain ring and the sprocket. If there are n_1 teeth in the chain ring and n_2 teeth in the sprocket then n_1/n_2 is the gear ratio. Suppose that the radii of the chain ring, sprocket and rear wheel are $r_1,\ r_2$ and r respectively and that the angular velocities of the chain ring and rear wheel are ω_1 and ω_2 respectively.



Bicycle drive system.

(a) Write down an expression for the velocity (u) of the bicycle in terms of the angular velocity of the rear wheel:

Your solution

Answer

$u = r\omega_2$

(b) Write down the relationship between the velocity of the sprocket and the velocity of the chain ring:

Your solution

Answer

$r_1\omega_1 = r_2\omega_2$

(c) Write down expressions for the circumference of the chain ring and that of the sprocket in terms of the teeth width and number of teeth:

Your solution

Answer

$2\pi r_1 = n_1 d,\ 2\pi r_2 = n_2 d$

(d) Hence derive a relationship between the angular velocities and the gear ratio:

Your solution

Answer

From (b) and (c), $\dfrac{n_1 d}{2\pi}\,\omega_1 = \dfrac{n_2 d}{2\pi}\,\omega_2$ or $n_1\omega_1 = n_2\omega_2$, so gear ratio $= \dfrac{\omega_2}{\omega_1}$

(e) Calculate the speed of the bicycle if the cyclist is pedalling at one revolution per second, the radius of the rear wheel is 0.34 m and the gear ratio is 4:

Your solution

Answer

From (a) and (d), $u = r(n_1/n_2)\omega_1 = 0.34(4)(2\pi) = 8.55$ m s^{-1} (about 19 mph).

2. Amusement rides

The design of amusement rides is intended to make the forces experienced by passengers as exciting as possible. High speeds are not enough. The production of accelerations of up to four times that due to gravity and occasional feelings of near-weightlessness are deliberate design goals. The accelerations may not only be in the forward or backward directions from the passengers' perspective but also sideways. Sideways accelerations are more limited than backwards or forwards ones, since they are less welcome to passengers and also pose particular problems for the associated structures. Upward accelerations of greater than g are avoided because of the safety risk and the need for reliable constraints to prevent passengers 'floating' out of the carriages. The rates at which the forces, or accelerations, change are important in producing the overall sensation also. The rate of change of acceleration is called **jerk**. Even the rate of change of jerk, called **jounce**, may be of interest. On roller-coasters the height, the tightness and the twistiness of turns in the track are the main parameters that influence the thrill of riding on them. Another factor that relates to thrill or discomfort is the degree of mismatch between what is observed and what is felt. For example it is odd to feel as though one's weight is acting in some other direction than the perceived vertical. In this regard the *magnitude* of the force that is experienced may not be as important as its *direction*. Although we consider only two dimensions in this Workbook, the design of roller-coasters requires calculations and considerations in 3D. We shall consider some particular examples of forces experienced by passengers. We start by exploring the forces that contribute to what we feel when riding in a vehicle. Then we shall look at the forces experienced by passengers on amusement rides ranging from rotors to roller-coasters.

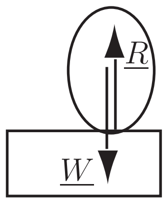


Figure 21: Forces on a seated passenger

Fearsome forces

The experiences we have on amusement rides include those of being subjected to linear and sideways accelerations that are sensed by the balancing system close to our ears and can make us feel giddy. Potentially more 'enjoyable' are sensations of unfamiliar compressive forces that act on our bodies through the vehicle in which we are travelling.

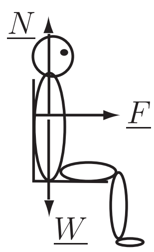


Figure 22: Forces when seated and being accelerated horizontally

Imagine that you are sitting on a stool, which is sufficiently tall so that your feet are not touching the ground. What do you feel? Your weight is acting vertically downward. However you feel an upward force, which is the normal reaction of the stool to your weight. This reaction is pushing you upward (see Figure 21). Of course the total force on you is zero. Consequently there is a difference in this case between the total force on you, which includes gravity, and the force that you experience which excludes gravity.

On the other hand if you are sitting facing forwards in a vehicle that is accelerating forwards on a flat track then you will experience the same acceleration as the vehicle, through the seat which pushes you forwards (see Figure 22). The forward force from the seat combines with the normal reaction to give a resultant that is not vertical. This simple example suggests that the force experienced by a passenger during an amusement ride can be calculated by adding up all the component forces except for the passenger's weight.

Example 11

(a) Calculate the horizontal and vertical components of the force $\underline{F}$ experienced by a passenger of mass 100 kg seated in a rollercoaster carriage that starts from rest and moves in a straight line on a flat horizontal track with a constant acceleration such that it is moving at 40 m s^{-1} after 5 s.

(b) Deduce the magnitude and direction of the force experienced by the passenger.

Solution

(a) Consider first the acceleration of the passenger's seat. The coordinate origin is chosen at the start of motion. The x-axis is chosen along the direction of travel, with unit vector $\underline{i}$, and the y-axis is vertical with y positive in the upward direction, with unit vector $\underline{j}$. The acceleration $\underline{a}$ may be calculated from

$$\underline{a} = \frac{d\underline{v}}{dt} = a\underline{i}$$

or $\underline{v} = at\underline{i} + \underline{c}$. When $t = 0, \underline{v} = \underline{0}$, so $\underline{c} = \underline{0}$. When $t = 5, \underline{v} = 40\underline{i}$. Hence $40 = 5a$ or $a = 8$. The acceleration in the direction of travel is 8 m s^{-2}, so the component of the force F in the direction of travel is given by $ma = 100 \times 8 \text{ N} = 800$ N. The seat exerts a force on the passenger that balances the force due to gravity i.e. the passenger's weight. So the vertical component of the force on the passenger is $mg = 100 \times 9.81 \text{ N} = 981$ N. Hence the total force experienced by the passenger may be expressed as

$$\underline{F} = 800\underline{i} + 981\underline{j}.$$

This is the total force exerted by the vehicle on the passenger. The force exerted by the passenger on the vehicle is the direct opposite of this i.e. $-800\underline{i} - 981\underline{j}$.

(b) The magnitude of the total force is $\sqrt{800^2 + 981^2}$ N ≈ 1300 N and the total force experienced by the passenger is at an angle with respect to the horizontal equal to $\tan^{-1}(981/800) \approx 51°$.

Here we considered the sudden application of a constant acceleration of 8 m s^{-2} which will cause quite a jerk for the passengers at the start. On some rides the acceleration may be applied more smoothly.

Example 12

Calculate the total force experienced by the passenger as a function of time if the horizontal component of acceleration of the vehicle is given by

$$\begin{cases} 8\sin\left(\dfrac{\pi t}{10}\right) & 0 \leq t \leq 5 \\ 8 & t > 5 \end{cases}$$

Solution

With this horizontal component of acceleration, the component of force experienced by the passenger in the direction of motion is (see Figure 23)

$$\begin{cases} 800\sin\left(\dfrac{\pi t}{10}\right) & \text{for} \quad 0 \leq t \leq 5 \\ 800 & \text{for} \quad t > 5 \end{cases}$$

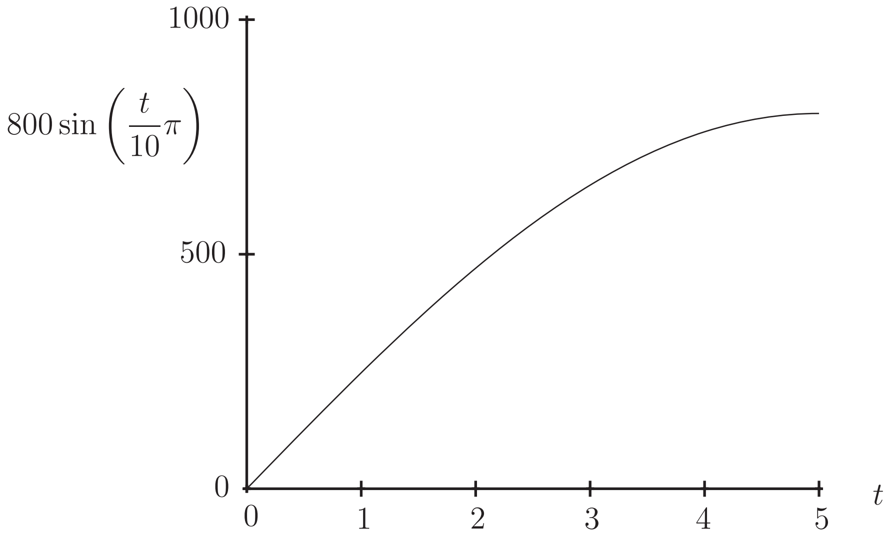


Figure 23: Horizontal component of force

The vertical component remains constant at 981 N, as in Example 11. The total force may be written

$$\underline{F} = \begin{cases} 800\sin\left(\dfrac{\pi t}{10}\right)\underline{i} + 981\underline{j} & 0 \leq t \leq 5 \\ 800\underline{i} + 981\underline{j} & t > 5 \end{cases}$$

Task

The idea of an amusement ride called the 'Rotor' is to whirl passengers around in a cylindrical container at increasing speed. When the rotation is sufficiently fast the floor is lowered but the passengers remain where they are supported by friction against the wall. Given a rotor radius of 2.2 m and a coefficient of friction of 0.4 calculate the minimum rate of revolution when the floor may be lowered.

Your solution

Answer

The reaction of the wall on the passenger will have the same magnitude as the force causing motion in a circle i.e. $\dfrac{mv^2}{r} = R$. The vertical friction force between the passenger and the wall is μR. The passenger of mass will remain against the wall when the floor is lowered if $\mu R \geq mg$. Hence it is required that $\dfrac{\mu m v^2}{r} \geq mg$ or $v \geq \sqrt{\dfrac{rg}{\mu}}$. Since $v = \omega r$, where ω is the angular velocity, the minimum required angular velocity is $\omega = \sqrt{\dfrac{g}{\mu r}}$ and the corresponding minimum rate of revolution $n = \dfrac{\omega}{2\pi}\sqrt{\dfrac{g}{\mu r}}$. Hence with $r = 2.2$ and $\mu = 0.4$, the rate of revolution must be at least 0.53 revs per sec or at least 32 revs per min.

Task

In an amusement ride called the 'Yankee Flyer', the passengers sit in a 'boat', which stays horizontal while executing a series of rotations on an arm about a fixed centre. Given that the period of rotation is 2.75 s, calculate the radius of rotation that will give rise to a feeling of near weightlessness at the top of each rotation.

Your solution

Answer

To achieve a feeling of 'near-weightlessness' near the top of the rotation, the force on the passenger towards the centre of rotation must be nearly equal and opposite to the reaction force of the seat on the passengers i.e. $\dfrac{mv^2}{r} = mg$. This means that $r = \dfrac{v^2}{g}$. Since $v = \omega r$, this requires that $r = \dfrac{g}{\omega^2}$. The period $T = \dfrac{2\pi}{\omega} = 2.75$ s, so $\omega = 2.285$ rad s^{-1}. Hence $r = 1.879$ m.

Example 13

An amusement ride carriage moves along a track at constant velocity in the horizontal (x-) direction. It encounters a bump of horizontal length L and maximum height h with a profile in the vertical plane given by

$$y(x) = \frac{h}{2}\left(1 - \cos\frac{2\pi x}{L}\right), \quad 0 \leq x \leq L.$$

Calculate the variation of the vertical component of force exerted on a passenger by the seat of the carriage with horizontal distance (x) as it moves over the bump.

Solution

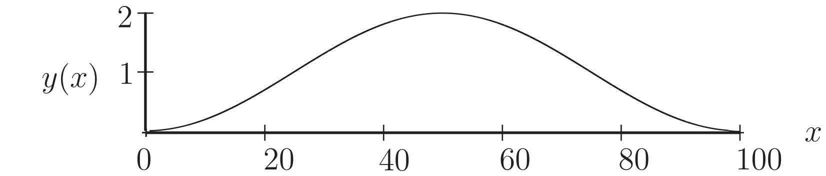


Figure 24: Profile of the bump

Figure 24 shows a graph of $y(x)$ against x for $h = 2$ and $L = 100$. Since the component of velocity of the vehicle in the x-direction is constant, then after time t, the horizontal distance moved, x, is given by $x = ut$ as long as x is measured from the location at $t = 0$. Consequently the y-coordinate may be written in terms of t rather than x, giving

$$y(t) = \frac{h}{2}\left(1 - \cos\frac{2\pi ut}{L}\right), \quad 0 \leq t \leq \frac{L}{u}.$$

The vertical component of velocity is given by differentiating this expression for $y(t)$ with respect to t.

$$\dot{y}(t) = \frac{h}{2}\frac{2\pi u}{L}\sin\left(\frac{2\pi ut}{L}\right) = \frac{h\pi u}{L}\sin\left(\frac{2\pi ut}{L}\right).$$

The vertical acceleration is given by differentiating this again.

$$\ddot{y}(t) = \frac{h\pi u}{L}\frac{2\pi u}{L}\cos\left(\frac{2\pi ut}{L}\right) = \frac{2h\pi^2u^2}{L^2}\cos\left(\frac{2\pi ut}{L}\right).$$

Two forces contribute to the vertical force exerted by the seat on the passenger: the constant reaction to the passenger's weight and the variable vertical reaction associated with motion over the bump. The magnitude of the total vertical force R N exerted on the passenger by the seat is given by

$$R = mg + m\ddot{y}(t) = mg + m\frac{2h\pi^2u^2}{L^2}\cos\left(\frac{2\pi ut}{L}\right) = mg\left(1 + \frac{2h\pi^2u^2}{gL^2}\cos\left(\frac{2\pi ut}{L}\right)\right).$$

Some horizontal force may be needed to keep the vehicle moving with a constant horizontal component of velocity and ensure that the net horizontal component of acceleration is zero. Since the horizontal component of velocity is constant, there is no horizontal component of acceleration and no net horizontal component of force. Consequently $\underline{R} = R\underline{j}$ represents the total force exerted on the passenger. Figure 25 shows a graph of R/mg for $h = 2$ m, $L = 100$ m and $u = 20$ m s^{-1}.

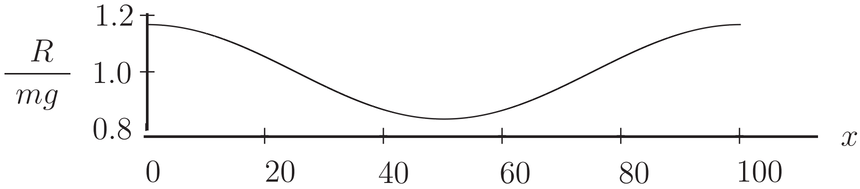


Figure 25: Vertical force acting on passenger

Banked tracks

Look back near the end of Section 34.2 subsection 1 which considered the forces on a cyclist travelling around a circular bend of radius R. We were concerned with the way in which cyclists and motorcyclists bank their vehicles to create a 'new vertical' along the direction of the resultant force. This counteracts the torque that would otherwise encourage the rider to fall over when cornering. Clearly, passengers in four wheeled vehicles, railway trains and amusement park rides are not able to bank or tilt their vehicles to any significant extent. However what happens if the road or track is banked instead? If the road or track is tilted or banked at angle $\theta = \tan^{-1}\left(\dfrac{v^2}{gr}\right)$ to the horizontal, then, at speed v around the circular bend, it is possible to obtain the same result as that achieved by tilting the cycle or motorcycle (see Figure 26).

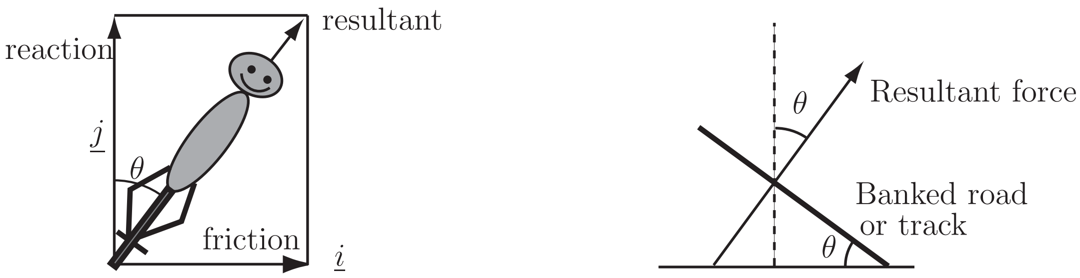


Figure 26: Equivalence of tilted cyclists and banked roads

Example 14

Calculate the angle at which a track should be tilted so that passengers in a railway carriage moving at a constant speed of 20 m s^{-1} around a bend of radius 100 m feel the resultant 'reaction' force as though it were acting vertically through their centre line.

Solution

The angle of the resultant force on passengers if the track were horizontal is given by

$$\theta = \tan^{-1}\left(\frac{v^2}{gr}\right),$$

where $v = 20, r = 100$ and $g = 9.81$.

Hence $\theta = 22.18°$ and the track should be tilted at $22.18°$ to the horizontal for the resultant force to act at right-angles to the track.

Example 15

The first three seconds of an amusement ride are described by the position coordinates

$$x(t) = 10 - 10\cos(0.5t) \qquad (0 < t < 3)$$
$$y(t) = 20\sin(0.5t)$$

in the horizontal plane where x and y are in m.

(a) Calculate the velocity and acceleration vectors.

(b) Hence deduce the initial magnitude and direction of the acceleration and the magnitude and direction of the acceleration after three seconds.

Solution

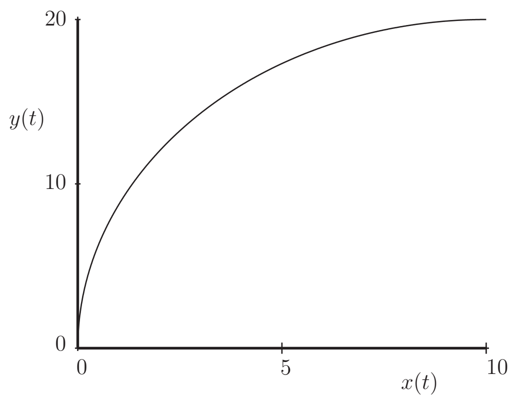


Figure 27: Path of ride

In this Example, the path of the ride is not circular (see Figure 27). In fact it is part of an ellipse. If we choose unit vectors $\underline{i}$ along the x-direction and $\underline{j}$ along the y-direction, and origin at $t = 0$, the position vector may be written

$$\underline{r}(t) = (10 - 10\cos(0.5t))\underline{i} + 20\sin(0.5t)\underline{j}.$$

The velocity vector is obtained by differentiating this with respect to time.

$$\underline{v}(t) = 5\sin(0.5t)\underline{i} + 10\cos(0.5t)\underline{j}.$$

The acceleration vector is obtained by differentiating again.

$$\underline{a}(t) = 2.5\cos(0.5t)\underline{i} - 5\sin(0.5t)\underline{j}$$

At $t = 0, \underline{a}(t) = 2.5\underline{i}$. So the initial acceleration is 2.5 m s^{-2} in the x-direction.

At $t = 3, \underline{a}(t) = 2.5\cos(1.5)\underline{i} - 5\sin(1.5)\underline{j} = 0.177\underline{i} - 0.487\underline{j}$.

So after three seconds the acceleration is 4.99 m s^{-2} at an angle of 88° in the negative y-direction. This means a sideways acceleration of about 0.5g towards the inside of the track and almost at right-angles to it.

Engineering Example 1

Car velocity on a bend

Problem in words

A road has a bend with radius of curvature 100 m. The road is banked at an angle of $10°$. At what speed should a car take the bend in order not to experience any (net) side thrust on the tyres?

Mathematical statement of the problem

Figure 28 below shows the forces on the car.

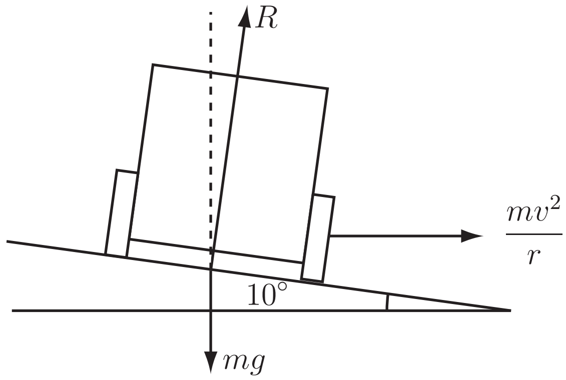


Figure 28: A vehicle rounding a banked bend in the road.

In the figure R is the reactive force of the ground acting on the vehicle. The vehicle provides a force of mg, the weight of the vehicle, operating vertically downwards. The vehicle needs a sideways force of $\frac{mv^2}{r}$ in order to maintain the locally circular motion.

We have used the following assumptions:

(a) The sideways force needed on the vehicle in order to maintain it in circular motion (called the centripetal force) is $\frac{mv^2}{r}$ where r is the radius of curvature of the bend, v is the velocity and m the mass of the vehicle.

(b) The only force with component acting sideways on the vehicle is the reactive force of the ground. This acts in a direction normal to the ground. (That is, we assume no frictional force in a sideways direction.)

(c) The force due to gravity of the vehicle is mg, where m is the mass of the vehicle and g is the acceleration due to gravity (≈ 9.8 m s^{-2}). This acts vertically downwards.

The problem we need to solve is 'What value of v would be such that the component of the reactive force of the ground exactly balances the sideways force of $\frac{mv^2}{r}$?' This will give us the maximum velocity at which the vehicle can take the bend.

Mathematical analysis

We can split the reactive force of the ground into two components. One component is in the horizontal direction and the other in a vertical direction as in the following figure:

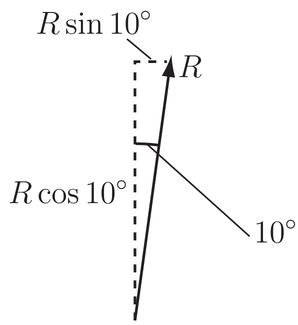


Figure 29: Reaction forces on the car

The force of $\dfrac{mv^2}{r}$ must be provided by a component of the reactive force in the horizontal direction i.e.

$$R\sin(10°) = \frac{mv^2}{r} \qquad (1)$$

However the reactive force must balance the force due to gravity in the vertical direction therefore

$$R\cos(10°) = mg \qquad (2)$$

We need to find v from the above equations. Dividing Equation (1) by Equation (2) gives

$$\tan(10°) = \frac{v^2}{rg} \Rightarrow v^2 = rg\tan(10°)$$

We are given that the radius of curvature is 100 m and that $g \approx$ 9.8 m s^{-2}. This gives

$v^2 \approx 100 \times 9.8 \times 0.17633$

$\Rightarrow \quad v^2 \approx 172.8$

$\Rightarrow \quad v \approx$ 13.15 m s^{-1} (assuming v is positive)

Interpretation

We have found that the maximum speed that the car can take the bend in order not to experience any side thrust on the tyres is 13.15 m s^{-1}. This is $13.15 \times 60 \times 60/1000$ kph $= 47.34$ kph. In practice, the need for a margin of safety would suggest that the maximum speed round the bend should be 13 m s^{-1}.

Exercises

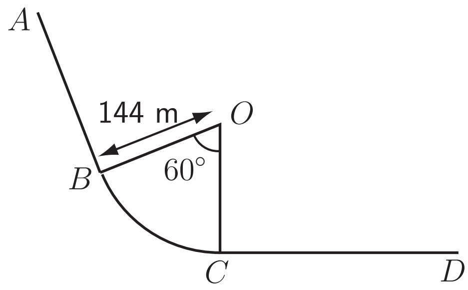
1. A bend on a stretch of railway track has a radius of 200 m. The maximum sideways force on the train on this bend must not exceed 0.1 of its weight.

   (a) What is the maximum possible speed of the train on this bend?

   (b) How far before this bend should a train travelling at 30 m s^{-1} begin to decelarate given that the maximum braking force of the train is 0.2 of its weight?

   (c) What modelling assumptions have you made? Comment on their validity.

2. The diagram shows a portion of track of a one-way fairground ride on which several trains are to run. AB and CD are straight. BC is a circular arc with the dimension shown. Because BC is also on a bridge, safety regulations require that the rear of one train must have passed point C before the front of the next train passes point B. Trains are 30 m long.



   If the maximum sideways force on a train can be no more than 0.1 of its weight, find the shortest time it can take for a train to travel from B to C. Hence find the minimum time between the front of one train passing point B and its rear end passing point C. Recommend a minimum distance between trains.

Answers

1. (a) According to Equation (2.2) on page 35, during travel round the bend the sideways force on the train is given by

$$Mr\omega^2 = \frac{Mv^2}{r}$$

The weight of the train is Mg. Given that $\frac{Mv^2}{r} \leq 0.1Mg$, the maximum possible speed, $v_{\max}$, is given by $v_{\max} = \sqrt{0.1rg}$.

Using $r = 200$ m and $g = 9.81$ m s^{-2}, this implies that $v_{\max} = 14.0$ m s^{-1} to 3 s.f.

(b) Given initial speed is 30 m s^{-1}, and final speed is 14 m s^{-1} and maximum braking force is 0.2 Mg, implying acceleration is $-0.2g$. Then, using the formula '$v^2 = u^2 + 2as$', where u is initial speed, v its final speed, a is acceleration and s is distance travelled, gives

$$20g = 900 - 0.4gs \qquad \text{or} \qquad s = 179.358 \text{ m.}$$

This suggests that braking should begin about 180 m before the start of the bend.

(c) Assumptions include constant maximum braking, negligible thinking time and no skidding.

2. The shortest time on the circular bend will be taken when the train is moving at the maximum possible speed.

This will occur when $\frac{Mv^2}{r} = 0.1Mg$. If $r = 144$ m and $g = 9.81$ m s^{-2}, this implies $v_{\max} = 11.885$ m s^{-1}.

The length of BC is $144\frac{\pi}{3} = 150.796$ m.

The time taken for any point on the train to move from B to C is $\frac{150.796}{11.885} = 12.687$ s.

So, given that the length of each train is 30 m, to make sure that the rear of one train has passed C before the front of the next train arrives at B, a minimum time between the trains of $(12.687 + \frac{30}{11.885})$ s $= 15.212$ s is required. After including a small safety margin, each train should be 16 s apart. Assuming that the trains are moving at a constant speed of 11.885 m s^{-1}, this implies that they should be at least 190 m apart.

Resisted Motion

34.3

Introduction

This Section returns to the simple models of projectiles considered in Section 34.1. It explores the magnitude of air resistance effects and the effects of including simple models of air resistance on the earlier analysis.

Prerequisites

Before starting this Section you should ...

- be able to solve second order, constant coefficient ODEs
- be able to use Newton's laws to describe and model the motion of particles

Learning Outcomes

On completion you should be able to ...

- compute the effect of air resistance proportional to velocity on particles moving under gravity
- define terminal velocity for linear and quadratic dependence of resistance on velocity

1. Resisted motion

Resistance proportional to velocity

In Section 34.2 we introduced methods of analysing the motion of projectiles on the assumption that air resistance or **drag** can be neglected. In this Section we will consider the accuracy of this assumption in some particular cases and take a look at the consequences which including air resistance has for the vector analysis of forces and motion.

Consider the subsequent motion of an object that is thrown horizontally. Let us introduce coordinate axes x (horizontal, unit vector $\underline{i}$) and y (vertical upwards, unit vector $\underline{j}$) and place the origin of coordinates at the point of release. The forces on the object consist of the weight $mg\underline{j}$ and a resisting force proportional to the velocity v. This force may be written

$$-c\underline{v} = -cx\underline{i} - c\dot{y}\underline{j},$$

where c is a constant of proportionality. Newton's second law gives

$$m\underline{a} = m(\ddot{x}\underline{i} + \ddot{y}\underline{j}) = (-c\dot{x}\underline{i} - c\dot{y}\underline{j} - mg\underline{j}).$$

This can be separated into two equations:

$$m\ddot{x} = -c\dot{x} \qquad (3.1)$$

and

$$m\ddot{y} = -c\dot{y} - mg. \qquad (3.2)$$

These equations each involve only one variable so they are **uncoupled**. They can be solved separately. Consider the Equation (3.1) for the horizontal motion, first in the form

$$m\ddot{x} + c\dot{x} = 0.$$

Dividing through by m and using a new constant $\kappa = c/m$,

$$\ddot{x} + \kappa\dot{x} = 0$$

A solution to this equation (see HELM 19) is

$$x = A + Be^{-\kappa t}$$

where A and B are constants. These constants may be evaluated by means of the initial conditions

$$x(0) = 0 \qquad \dot{x}(0) = v_0$$

where v_0 is the speed with which the object is thrown (recall that it is thrown horizontally). The first condition gives

$$0 = A + B$$

which means that $A = -B$. The second gives

$$v_0 = -B\kappa$$

which implies that $B = -\dfrac{v_0}{\kappa}$, so

$$x(t) = \frac{v_0}{\kappa}\left(1 - e^{-\kappa t}\right) \qquad (3.3)$$

The initial conditions for the vertical motion are

$$y(0) = 0 \qquad \dot{y}(0) = 0.$$

Equation (3.2), in the form

$$\ddot{y} + \kappa\dot{y} = -g$$

may be solved by multiplying through by $e^{\kappa t}$ (HELM 19) which enables us to write

$$\frac{d}{dt}\left(\dot{y}e^{\kappa t}\right) = -ge^{\kappa t}.$$

After integrating with respect to t twice,

$$y(t) = C + De^{-\kappa t} - \frac{gt}{\kappa}.$$

The initial conditions give

$$0 = C + D \qquad \text{and} \qquad 0 = -\kappa D - g/\kappa$$

which means that $D = -g/\kappa^2$, so $C = g/\kappa^2$ and

$$y(t) = \frac{g}{\kappa^2}(1 - e^{-\kappa t}) - \frac{gt}{\kappa}. \tag{3.4}$$

From Equation (3.1), the horizontal component of velocity is

$$\dot{x}(t) = v_0 e^{-\kappa t}. \tag{3.5}$$

The air resistance causes the horizontal component of velocity to decrease exponentially from its original value. From Equation (3.2), the upward vertical component of velocity is

$$\dot{y}(t) = \frac{g}{\kappa}\left(e^{-\kappa t} - 1\right). \tag{3.6}$$

For very large values of t, $e^{-\kappa t}$ is near zero, so the vertical component of velocity is nearly constant at $-g/\kappa$. The negative sign indicates that the object is moving downwards. g/κ represents the terminal velocity for vertical motion under gravity for a particle subject to air resistance proportional to velocity. Sketches of the variations of the components of velocity with time are shown in Figure 30.

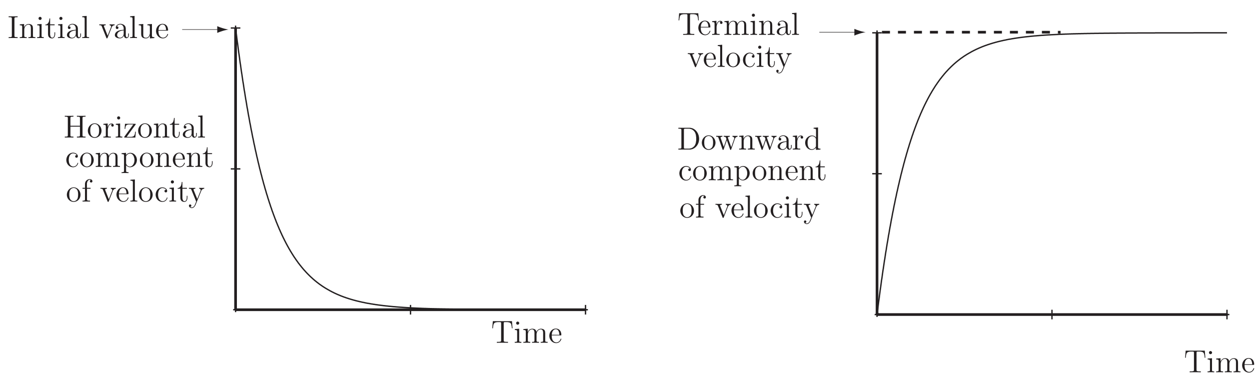


Figure 30
Velocity components of an object launched horizontally and subject to resistance proportional to velocity

By combining the components of velocity given in (3.5) and (3.6), it is possible to obtain the magnitude and direction of the velocity of an object projected horizontally at speed v_0 and subject to air resistance proportional to velocity, the magnitude is $\sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}$ and the direction is $\tan^{-1}(\dot{y}(t)/\dot{x}(t))$.

Note that the expression for terminal velocity could be obtained directly from (3.2), by setting $\ddot{y} = 0$.

Example 16

At the time that the parachute opens a parachutist of mass 100 kg is travelling horizontally at 20 m s^{-1} and is 200 m above the ground. Calculate (a) the parachutist's height above the ground and (b) the magnitude and direction of the parachutist's velocity after 10 s assuming that air resistance during the first 100 m of fall may be modelled as proportional to velocity with constant of proportionality $c = 100$.

Solution

(a) Substituting $m = 100, g = 9.81$ and $c = 100$ in Equation (3.4) gives the distance dropped during 10 s as 88.3 m. So the parachutist will be 111.7 m above the ground after 10 s. The model is valid up to this distance.

(b) The vertical component of velocity after 10 s is given by Equation (3.6) i.e. 9.81 m s^{-1}. The horizontal component of velocity is given by Equation (3.5) i.e. 9.08×10^{-4} m s^{-1}, which is practically negligible. So, after 10 s, the parachutist will be moving more or less vertically downwards at 9.81 m s^{-1}.

If the object is launched at some angle θ above the horizontal, then the initial conditions on velocity are

$$\dot{x}(0) = v_0 \cos\theta \qquad \dot{y}(0) = v_0 \sin\theta$$

These lead to the following equations, replacing (3.3) and (3.4):

$$x(t) = \frac{v_0 \cos\theta}{\kappa}\left(1 - e^{-\kappa t}\right) \tag{3.7}$$

$$y(t) = \left[\frac{v_0 \sin\theta}{\kappa} + \frac{g}{\kappa^2}\right]\left(1 - e^{-\kappa t}\right) - \frac{gt}{\kappa}. \tag{3.8}$$

To obtain the trajectory of the object, (3.7) can be rearranged to give

$$\left(1 - e^{-\kappa t}\right) = \frac{\kappa x}{v_0 \cos\theta} \quad \text{and} \quad t = -\frac{1}{\kappa}\ln\left(1 - \frac{\kappa x}{v_0 \cos\theta}\right).$$

These can be substituted in (3.8) to give

$$y = x\left(\tan\theta + \frac{g}{\kappa v_0 \cos\theta}\right) + \frac{g}{\kappa^2}\ln\left(1 - \frac{\kappa x}{v_0 \cos\theta}\right). \tag{3.9}$$

Figure 31 compares predictions from this result with those predicted from the result obtained by ignoring air resistance (Equations (3.1) and (3.2)). The effect of including air resistance is to change the projectile trajectory from a parabola, symmetrical about the highest point, to an asymmetric curve, resulting in reduced maximum range.

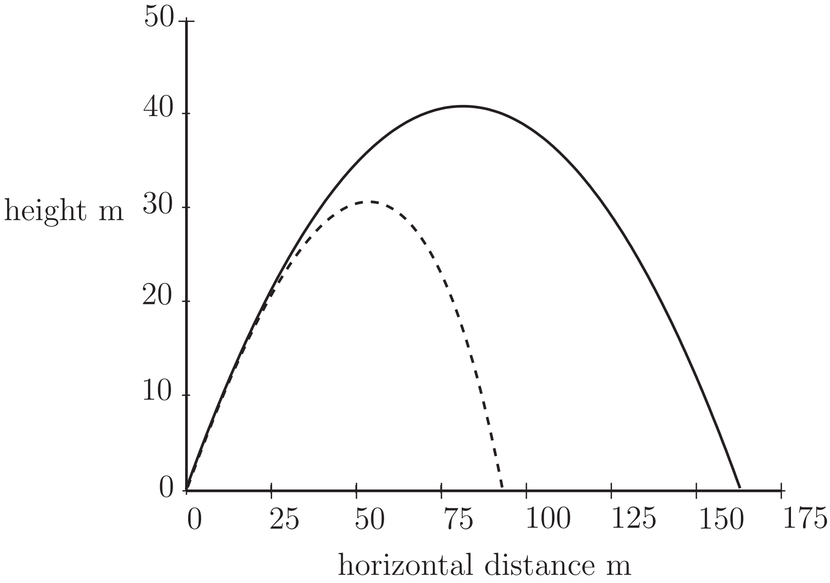


Figure 31

Predicted trajectories of an object projected at 45° with speed 40 m s^{-1} in the absence of air resistance (solid line) and with air resistance proportional to velocity such that $\kappa = 0.184$ (broken line)

Quadratic resistance

Unfortunately, it is not often very accurate to model air resistance by a force that is simply proportional to velocity. For a spherical object, a good approximation for the dependence of the air resistance force vector $\underline{R}$ on the speed ($\underline{v}$) and diameter (D) of the object is

$$\underline{R} = (c_1 D + c_2 D^2 |\underline{v}|)\underline{v} \qquad (3.10)$$

with $c_1 = 1.55 \times 10^{-4}$ and $c_2 = 0.22$ in SI units for air. As would be expected intuitively, the bigger the sphere and the faster it is moving the greater the drag it will experience. If D and $|\underline{v}|$ are very small then the second term in (3.10) can be neglected compared with the first and the **linear approximation** is reasonable i.e.

$$\underline{R} \simeq c_1 D \underline{v} \qquad D|\underline{v}| \leq 10^{-5}. \qquad (3.11)$$

Note that $c_1 << c_2$, so if D and $|\underline{v}|$ are not very small, for example a cricket ball ($D = 0.7$ m) moving at 40 m s^{-1}, the first term in (3.10) can be neglected compared with the second. This gives rise to the **quadratic approximation**

$$\underline{R} \simeq c_2 D^2 |\underline{v}|\underline{v} \qquad 10^{-2} \leq D|\underline{v}| \leq 1. \qquad (3.12)$$

The ranges of validity of these approximations are shown graphically in Figure 32 for a sphere of diameter 0.01 m. In general the linear approximation is accurate for small slow-moving objects and the quadratic approximation is satisfactory for larger faster objects. The linear approximation is similar to Stokes' law (first stated in 1845):

$$|\underline{R}| = 6\pi\mu r|\underline{v}| \qquad (3.13)$$

where μ is the coefficient of viscosity of the fluid surrounding a sphere of radius r. According to Stokes' law, $c_1 = 3\pi\mu$. This gives $c_1 = 0.17 \times 10^{-4}$ kg m^{-1}s^{-1} for air. Similarly, the quadratic approximation is consistent with a relationship deduced by Prandtl (first stated in 1917) for a sphere:

$$|\underline{R}| = 0.625\rho r^2 |\underline{v}|^2 \tag{3.14}$$

where ρ is the density of the fluid. This implies that $c_2 = 0.625\rho/4 = 0.202$ kg m^{-3} for air.

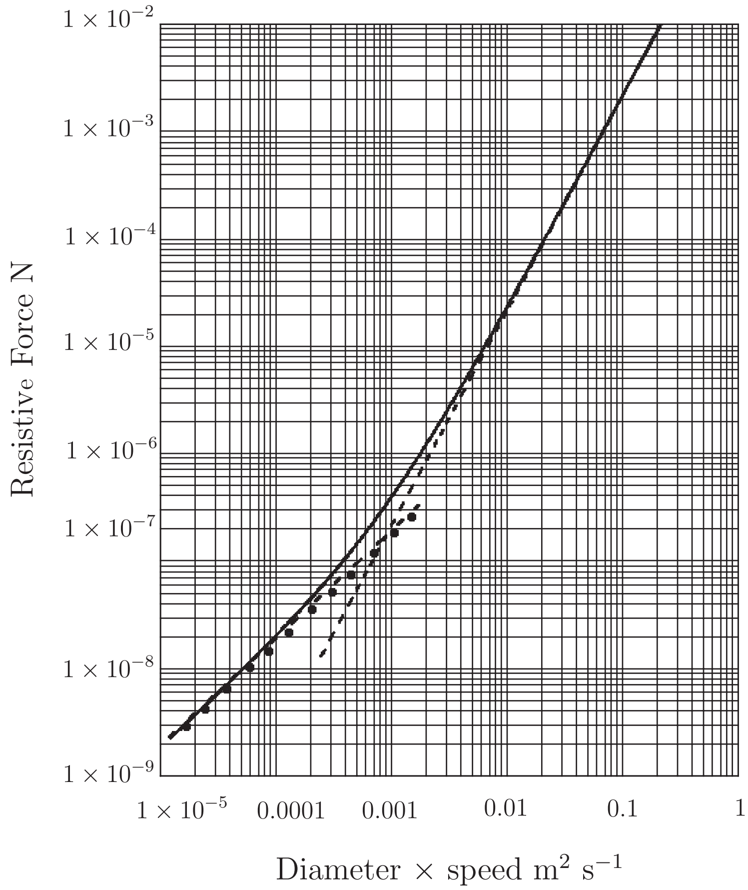


Figure 32

Resistive force, as a function of the product of diameter and speed, predicted by Equation (3.10) (solid line) and the approximations Equation (3.11) (broken line) and Equation (3.12) (dash-dot line), for a sphere of diameter 0.01m

The mathematical complexity of the equations for projectile motion in 2D resulting from the quadratic approximation is considerable. Consider an object with an initial horizontal velocity and the same coordinate axes as before, but this time the resistive force is given by $c|\underline{v}|\underline{v}$ (the quadratic approximation). For this case Newton's second law gives

$$m\underline{a} = m(\ddot{x}\underline{i} + \ddot{y}\underline{j}) = -c\dot{x}\sqrt{(\dot{x}^2 + \dot{y}^2)}\underline{i} - c\dot{y}\sqrt{(\dot{x}^2 + \dot{y}^2)}\underline{j} - mg\underline{j}.$$

The corresponding scalar differential equations are

$$m\ddot{x} = -c\dot{x}\sqrt{(\dot{x}^2 + \dot{y}^2)}$$

and

$$m\ddot{y} = -c\dot{y}\sqrt{(\dot{x}^2 + \dot{y}^2)} - mg.$$

You should note that $\dot{x}$ and $\dot{y}$ appear in both equations and cannot be separated out. These differential equations are **coupled**. (Ways of dealing with such coupled equations is introduced in HELM 20.)

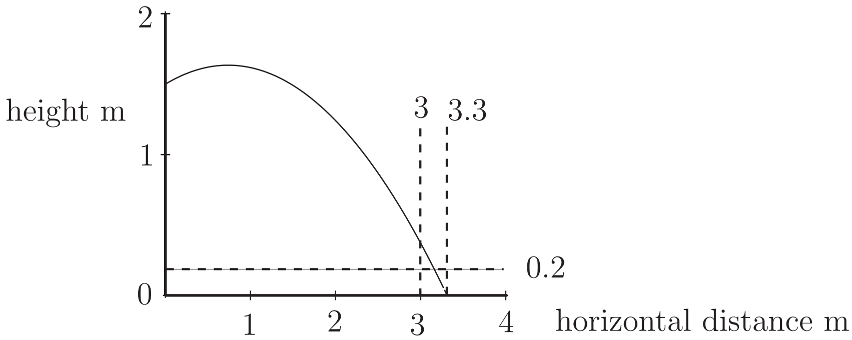
Task

Suppose that the academic in Example 1.5 screws up sheets of paper into spheres of radius 0.03 m and mass 0.01 kg. Calculate the effect of linear air resistance on the likelihood of the chosen trajectory entering the waste paper basket.

Your solution

Answer

Since $D|\underline{v}| = 4.75 \times 0.06 = 0.285$, the linear approximation for air resistance is not valid. If however it is assumed that it is, then $\kappa = c_1 D/m = 1.55 \times 10^{-4} \times 0.06/0.01 = 9.3 \times 10^{-4}$. A plot of the resulting trajectory according to Equation (3.9) is shown in the diagram below.



Predicted trajectory of paper balls with linear air resistance

With the stated assumptions, air resistance is predicted to have little or no effect on the trajectory of the paper balls.

Vertical motion with quadratic resistance

Although it is not straightforward to model motion in 2D with resistance proportional to velocity squared, it is possible to consider the motion of an object falling vertically under gravity experiencing quadratic air resistance. In this case the equation of motion may be written in terms of the (vertical) velocity ($\dot{y} = v$) as

$$m\frac{dv}{dt} = mg - cv^2.$$

This nonlinear differential equation can be solved by using separation of variables (HELM 19). First we rearrange the differential equation to give

$$\frac{dt}{dv} = \frac{-m/c}{-\dfrac{mg}{c} + v^2}.$$

Then we integrate both sides with respect to v, and write $\kappa_1 = c/m$ (note that this c is different from the c used for linear air resistance) which yields

$$t + C = \frac{1}{2\sqrt{g\kappa_1}} \ln\left(\frac{a+v}{a-v}\right)$$

where $a = \sqrt{g/\kappa_1}$. If the object starts from rest $C = 0$, so

$$\frac{a+v}{a-v} = e^{2t\sqrt{g\kappa_1}} \text{ and}$$

$$v = a\frac{\left(1 - e^{-2t\sqrt{g\kappa_1}}\right)}{\left(1 + e^{-2t\sqrt{g\kappa_1}}\right)} = a\tanh(t\sqrt{g\kappa_1}). \qquad (3.15)$$

Note that for $t \to \infty$ this predicts that the terminal velocity $v_t = a = \sqrt{g/\kappa_1}$. This expression for terminal velocity may be compared with that for linear air resistance (g/κ). So the quadratic resistance model predicts a square root form for terminal velocity. Note that the expression for the terminal velocity for vertical motion of a particle subject to resistance proportional to the square of the velocity could be obtained from $m\dfrac{dv}{dt} = mg - cv^2$ by setting $\dfrac{dv}{dt} = 0$. If we write $\tau = \dfrac{v_t}{g}$ (note that this has units of time), then Equations (3.6) and (3.15) may be written

$$v = v_t\left(1 - e^{-t/\tau}\right)$$

and

$$v = v_t\frac{(1 - e^{-2t/\tau})}{(1 + e^{-2t/\tau})}.$$

Using these expressions, it is possible to compare the variation of the ratio v/v_T as a function of time in units of τ as in Figure 33. The graph shows the intuitive result that a falling object subject to quadratic resistance approaches its terminal velocity more rapidly than a falling object subject to resistance proportional to velocity. For example, at $t/\tau = 5, v/v_t$ is 0.993 with linear resistance and 0.9991 with quadratic resistance. Note however that the terminal velocities and the time steps used in the graph are different.

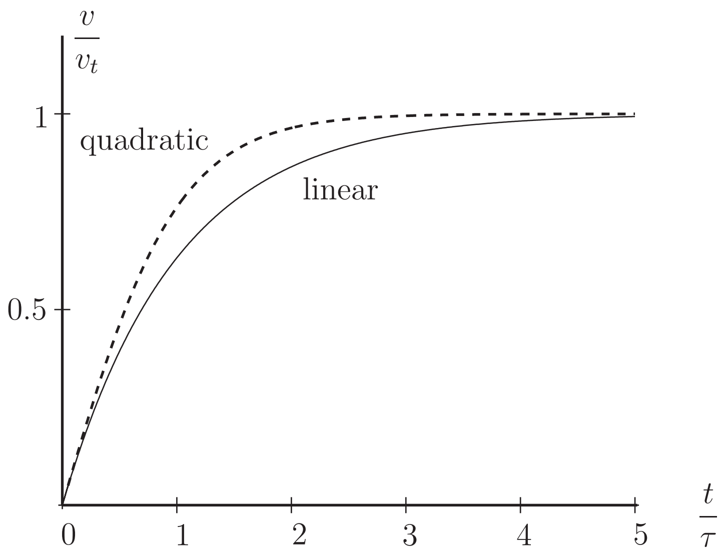


Figure 33
Comparison of the variations in vertical velocities
for a falling object subject to linear and quadratic resistance

Task

Note that the curves in Figure 33 are very close to each other and almost straight for small values of t/τ. Why should this be the case? As well as proposing an intuitive explanation, consider the result of expanding the exponential term in (3.6) in a Maclaurin power series.

Your solution

Answer
It is to be expected that, in the initial stages of motion when v and t are small, the gravitational force will dominate over air resistance i.e. $v \approx -gt$. A Maclaurin power series expansion of the exponential term in (3.6) gives

$$e^{-\kappa t} = 1 - \kappa t + \frac{1}{2}(\kappa t)^2 - \ldots$$

So $\quad \mathrm{e}^{-kt} - 1 = -\kappa t + \frac{1}{2}(\kappa t)^2 - \ldots \quad$ so $\quad v \approx \frac{g}{\kappa} \times (-\kappa t + \frac{1}{2}(\kappa t)^2 - \ldots)$

If t is much smaller than $1/\kappa$, then only the first term need be considered which gives $v \approx -gt$.